

**TOPICS IN SPATIAL AND DYNAMICAL PHASE TRANSITIONS OF
INTERACTING PARTICLE SYSTEMS**

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TOPICS IN SPATIAL AND DYNAMICAL PHASE TRANSITIONS OF INTERACTING PARTICLE SYSTEMS

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To Alejo and María

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LIST OF SYMBOLS

Assorted notation

$O(g)$	Big O notation.	3
$\Omega(g)$	Big Ω notation.	3
$o(g)$	Small o notation.	3
$\omega(g)$	Small ω notation.	3

Gibbs measure

$\gamma = (\psi, \phi)$	Gibbs specification.	5
$\mathcal{Z}_G(\cdot)$	Counting measure.	5
\mathcal{Z}_G	Partition function.	5
$\mu_G(\cdot)$	Normalized counting measure.	5
X	Set of spins.	5
$\psi_v(x)$	Self-interaction of the spin x at the vertex v	5
$\phi_e(x, y)$	Interaction of the spins x and y at the edge e	5
Ω_G	Set of all configurations $\sigma \in X^G$	5

Gibbs measures over trees

Γ_h	Boundary condition on the leaves of a tree.	18
$\Gamma_h^{(v)}$	Boundary condition restricted to the subtree subtended at v	19
$\psi_v^{(\Gamma_h)}$	Redefinition of the potential according to the boundary condition Γ_h	18
$\mu_{\mathbf{T}_h, \Gamma_h}(\cdot)$	Gibbs measure over \mathbf{T}_h with boundary condition Γ_h	19

$\mu_h(\cdot)$	Short for $\mu_{T_h, \Gamma_h}(\cdot)$	19
$\mu_{\mathbf{T}_h^{(v)}, \Gamma_h^{(v)}}(\cdot)$	Gibbs measure over $T_h^{(v)}$ with boundary condition $\Gamma_h^{(v)}$	19
$\mu_{h,v}(\cdot)$	Short for $\mu_{T_h^{(v)}, \Gamma_h^{(v)}}(\cdot)$	19
$\nu(\cdot)$	Simple invariant Gibbs measure.	22
$\nu_h(\cdot)$	Simple invariant Gibbs measure restricted to T_h	22
ssm	Rate of strong spatial mixing.	56
gssm	Rate of generalized strong spatial mixing.	58
Graph notation		
$d(\cdot, \cdot)$	graph distance.	4
$L_k(v)$	Vertices at distance $= k$ from the vertex v	4
$L_{\leq k}(v)$	Vertices at distance $\leq k$ from the vertex v	4
$L_{\geq k}(v)$	Vertices at distance $\geq k$ from the vertex v	4
$\mathcal{P}_{G,v}^{(i)}$	Path-tree of the graph G starting at v	54
Markov chain mixing time		
\mathcal{K}	Markov kernel of a markov chan.	7
$\tau_{\mathbf{mix}}(x, \epsilon)$	Mixing time.	7
$c_{\mathbf{gap}}$	Spectral gap.	8
$\tau_{\mathbf{rel}}$	Relaxation time.	8
Φ_S	Conductance of the set S	9
$c_{\mathbf{sob}}$	Log-sobolev constant.	37

Probability notation

$\mathbf{I}(\cdot)$	Indicator function.	3
\sim	‘distributed as ’.	3
$X \succsim Y$	Stochastic dominance.	4

Thresholds

uniq	Uniqueness threshold.	12
rec	Reconstruction threshold.	23

Trees

T	Infinite regular tree.	18
T_h	Regular tree of height h	18
T_h^(v)	Subtre subtended at the vertex v	18
L_k	Vertices at distance k from the root of the tree.	18
b	Branching of a tree.	18
\mathcal{F}_M	Class of finite trees with branching M	65
$\mathcal{F}_{\leq M}$	Completion of \mathcal{F}_M under the ‘induced subtree’ relation.	66

SUMMARY

In this work we provide several improvements in the study of phase transitions of interacting particle systems:

- We determine a quantitative relation between non-extremality of the limiting Gibbs measure of a tree-based spin system, and the temporal mixing of the *Glauber Dynamics* over its finite projections. We define the concept of ‘sensitivity’ of a reconstruction scheme to establish such a relation. In particular, we focus on the independent sets model, determining a phase transition for the mixing time of the Glauber dynamics at the same location of the extremality threshold of the simple invariant Gibbs version of the model.
- We develop the technical analysis of the so-called *spatial mixing conditions* for interacting particle systems to account for the connectivity structure of the underlying graph. This analysis leads to improvements regarding the location of the *uniqueness/non-uniqueness* phase transition for the independent sets model over amenable graphs; among them, the elusive hard-square model in lattice statistics, which has received attention since Baxter’s solution of the analogous hard-hexagon in 1980 [8].
- We build on the work of Montanari and Gerschenfeld [43] to determine the existence of correlations for the coloring model in sparse random graphs. In particular, we prove that correlations exist above the ‘clustering’ threshold of such a model; thus providing further evidence for the conjectural algorithmic ‘hardness’ occurring at such a point.

Keywords: Lattice gas, Gibbs measures, Phase transition, Approximation Algorithm, Glauber Dynamics, Constraint Satisfaction Problem, Reconstruction, Extremality of Gibbs measures, Uniqueness of Gibbs Measures, Spatial Mixing, Coloring, Random graphs.

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CHAPTER I

INTRODUCTION

The study of interacting particle systems is a large and growing field of probability theory devoted to the rigorous understanding of certain types of models that arise in combinatorics, statistical physics, biology, economics, and other fields. An interacting particle system can be loosely described as a set of distinguishable particles arranged in a fixed graphical configuration (lattice, tree, random graph, etc.), where every particle takes a certain value or ‘spin’ (orientation, color, charge), constituting a configuration of the system. The important feature of this object is the fact that the particles ‘interact’ among the edges of the graph, in a way prescribed by a given physical *specification*. Such a feature gives origin to a complex system in which the local interplay among particles is finally reflected into global observables of the system. In fact, the relation between local properties of the system and global observables is a fundamental notion in the theory of interacting particle systems. A *phase transition*, for a given parameter of the system, dictates the critical point at which such relation starts happening: the point at which a small change in the parameter causes a dramatic change in the macroscopic properties of the system. Examples of such behaviour are the so-called critical inverse temperature β_c for the Ising or the Potts model, and the critical activity λ_c for the independent sets model.

The occurrence of phase transitions is intimately related to the computational complexity of estimating the partition function \mathcal{Z} of the system. Recently, a remarkable connection was established between the computational complexity of approximating the partition function of the independent sets model for graphs of maximum degree Δ and the phase transition $\lambda_{\text{uniq}}(T)$ for the infinite regular tree T of degree Δ . On the positive side, Weitz [114] showed a deterministic fully-polynomial time approximation algorithm (FPAS) for approximating the partition function for any graph with maximum degree Δ , when $\lambda < \lambda_{\text{uniq}}(T)$. On the other side, Sly [102] recently showed that, for every $\Delta \geq 3$, it is NP-hard (unless

$\text{NP}=\text{RP}^1$) to approximate the partition function for graphs of maximum degree Δ , when $\lambda_{\text{uniq}}(T) < \lambda < \lambda_{\text{uniq}}(T) + \epsilon_\Delta$, for some function $\epsilon_\Delta > 0$ ².

Phase transitions are also thought to be linked with the mixing time of Markov chains arising from single-site updates (known as Glauber dynamics) for sampling interacting particle systems on finite graphs. The Glauber dynamics is well-studied both for its computational purposes, most immediately its use in Markov chain Monte Carlo (MCMC) algorithms, and for its physical motivation as a model of how physical systems reach equilibrium. Several works in this topic focus on exploring the dynamical and spatial connections between the mixing time and equilibrium (or ‘spatial’) properties of the interacting particle system. The first of these (conjectural) connections relates the *uniqueness* phase transition with the mixing time of Glauber dynamics on general graphs. The second relates the *extremality*, or *reconstruction* phase transition with the mixing time on trees and sparse random graphs.

The objectives achieved in this thesis are the following:

- Determining a quantitative relation between non-extremality of the limiting Gibbs’ measure of a tree-based interacting particle system, and the temporal mixing of the *Glauber Dynamics* over its finite projections. In particular, we focus on the independent sets model, establishing a phase transition for the mixing time of the Glauber dynamics at the same location of the extremality threshold of the simple invariant Gibbs version of the model.
- Developing the technical analysis of the so-called *spatial mixing conditions* on interacting particle systems to account for the connectivity structure of the underlying graph. This analysis leads to improvements regarding the location of the *uniqueness/non-uniqueness* phase transition for the independent sets model on amenable graphs. In particular, we obtain improved results regarding uniqueness of the independent sets model on \mathbb{Z}^2 (also called *hard-square* model in lattice statistics) which has received attention since Baxter’s solution of the analogous hard-hexagon in 1980 [8].

¹NP and RP are the well known complexity classes ‘Non-deterministic polynomial time’ and ‘Randomized polynomial time’, respectively.

²More recently, Galanis et al. [38] improved the range of λ in Sly’s inapproximability result, extending it to all $\lambda > \lambda_{\text{uniq}}(T)$ for the cases $\Delta = 3$ and $\Delta \geq 6$.

- Determine the existence or not of non-vanishing long-range correlations for interacting particle systems defined on sparse random graphs. We build on the work of Montanari and Gerschenfeld [43] to answer such a question in the case of the coloring model in Erdos-Renyi random graphs $G(n, p)$ graphs with $p = \alpha/n$. In particular, we prove the existence of long range correlations above the *clustering threshold* (namely, in the presence of clustering) of the model. Thus, providing further evidence for the conjectural algorithmic ‘hardness’ occurring at such a point.

1.1 Preliminaries

1.1.1 Some notation

Analytic notation. Ocasionally, we will use the Landau notation to describe asymptotic behaviour. This is defined as follows:

- $f \sim g$ (*asymptotical equivalence*) iff $\lim f/g = 1$.
- $f = O(g)$ iff $\limsup f/g \leq C$ for some constant C .
- $f = \Omega(g)$ iff $\liminf f/g \geq c$, for some constant c .
- $f = o(g)$ iff $\limsup f/g = 0$.
- $f = \omega(g)$ iff $\liminf f/g = \infty$.

Also, we use the symbol \approx for ‘imprecise’ numerical statements (like $e \approx 2.7182$, or $\frac{b^b}{(b-1)^{b+1}} \approx \frac{e}{b}$). The letters C and c will always denote constants. We explicitly describe the parameters on which they may depend, if that is the case. For instance, $C_{\delta, b}$ is a numerical constant that depends only on δ and b , and c is just a fixed real number. When several constants must be included, we distinguish them by prime symbols, C , C' , c , c' and the like.

Probability notation. The indicator function is denoted by $\mathbf{I}(\cdot)$. Also, we use \sim to denote that a given random variable is distributed according to a certain probability measure. For instance, $X \sim \mu$ means ‘let X be a random variable with distribution μ ’. The notation $\stackrel{d}{=}$ expresses equality in distribution among random variables, i.e. $X \stackrel{d}{=} Y$.

We say that two events $A, B \subseteq \text{supp } \mu$, are μ -positively correlated if $\mu(A \cap B) \geq \mu(A)\mu(B)$.

Let \leq be a partial order for the set Ω , and let X and Y be random variables taking values in Ω . We say that X dominates stochastically Y , or $X \succsim Y$, if

$$\mathbf{P}(X \geq a) \geq \mathbf{P}(Y \geq a), \text{ for every } a \in \Omega. \quad (1)$$

We assume that the reader is familiar with standard probabilistic notions like coupling, Markov processes, total variation, Galton-Watson processes, product measures, etc. In particular, we denote the total variation distance between two probability measures μ and μ' by

$$\text{TV}(\mu, \mu') := \frac{1}{2} \sum_{\sigma} |\mu(\sigma) - \mu'(\sigma)|,$$

where, in the summation, σ runs over $\text{supp } \mu \cup \text{supp } \mu'$.

Graph notation. A graph $G = (V, E)$ consists of a set of vertices V and a set $E \subseteq V^2$ that induces an *adjacency relation* on V , which is denoted by \sim ³. Such relation, consequently, induces a distance on the vertices of G , that we denote by $d(\cdot, \cdot)$:

$$d(v, v') := \min \{k : \exists v_i \in V \text{ such that } v \sim v_1 \sim \dots \sim v_{k-1} \sim v'\}.$$

Given a vertex $v \in V$, we define $L_k(v)$, $L_{\leq k}(v)$ and $L_{\geq k}(v)$ as follows:

$$\begin{aligned} L_k(v) &:= \{v' \in V : d(v, v') = k\}, \\ L_{\geq k}(v) &:= \{v' \in V : d(v, v') \geq k\}, \\ L_{\leq k}(v) &:= \{v' \in V : d(v, v') \leq k\}. \end{aligned}$$

We advice the reader that we use the expressions $v \in V$ and $v \in T$ interchangeably. Same for $e \in E$ and $e \in G$. A *tree* is a connected acyclic graph. A *rooted tree* is a tree with a distinguished vertex r called the *root*. Given a rooted tree T , we define

$$\begin{aligned} L_k &:= \{v \in T : d(v, r) = k\}, \\ L_{\geq k} &:= \{v \in T : d(v, r) \geq k\}. \end{aligned}$$

³Notice the three different uses of the symbol \sim . However, in different contexts.

1.1.2 Interacting particle systems

Let us consider a **finite** graph $G = (V, E)$ and a set X . A *Gibbs specification* $\gamma = (\psi, \phi)$ over G , with spin-set X , is a set of *self-interactions* $\psi_v(x)$ with $x \in X$, $v \in V$ and *interactions* $\phi_e(x, y)$ with $x, y \in X$, $e \in E$. The set of *configurations* of the system is the set Ω_G , consisting of all the configurations (i.e. *assignments*) $\sigma : G \rightarrow X$. The *counting measure* associated with the specification γ is the measure $Z_G(\cdot)$ defined on Ω_G such that for every $\sigma \in \Omega_G$,

$$Z_G(\sigma) = \prod_{v \in V} \psi_v(\sigma(v)) \prod_{e=(v,w) \in E} \phi_e(\sigma(v), \sigma(w)).$$

The corresponding Gibbs (or Boltzmann) measure is the equivalent probability distribution $\mu_G(\cdot)$, such that

$$\mu_G(\sigma) = \frac{Z_G(\sigma)}{Z_G},$$

where Z_G is the *partition function*

$$Z_G := \sum_{\sigma \in \Omega_G} Z_G(\sigma)$$

A specification is *uniform* if for all $v, v' \in V$ and $e, e' \in E$, $\phi_e = \phi_{e'}$ and $\psi_v = \psi_{v'}$. A q -spin system is a configuration such that $|X| = q$.

An *interacting particle system*, or *spin system* consists of a finite graph $G = (V, E)$ together with a given specification γ . The Gibbs' measure defined by the specification over G is denoted by μ_G .

For clarity, we prefer the use of the following probabilistic notation. Given a measure μ , let σ be a random variable distributed as μ (that is, $\sigma \sim \mu$). Then,

$$\mathbf{P}_{\sigma \sim \mu}(\sigma \text{ satisfies the property } P) := \mu(\{\eta \in \text{supp } \mu : \eta \text{ satisfies the property } P\}).$$

In particular,

$$\mathbf{P}_{\sigma \sim \mu_G}(\sigma \text{ satisfies the property } P) := \mu_G(\{\eta \in \Omega_G : \eta \text{ satisfies the property } P\}).$$

If $A \subseteq V$, and $\eta \in \Omega_G$, we denote by η_A the configuration η restricted to the set A . Also, we denote by μ_A the Gibbs' measure defined over G_A (the subgraph induced by A),

according to the specification γ restricted to G_A . Notice that if $\sigma \sim \mu_G$, then not necessarily is the case that $\sigma_A \sim \mu_{G_A}$.

Given $A \subseteq V$, let $\partial A := \{v \in V : v \sim A, v \notin A\}$. Now, given $\xi \in \Omega_{\partial A}$, we denote by μ_A^ξ , the Gibbs' measure defined on A with *boundary condition* ξ . μ_A^ξ is defined as follows:

$$\mu_A^\xi(\eta) := \mu_G(\{\eta' \in \Omega_G : \eta'_{\partial A} = \xi, \eta'_A = \eta\}), \quad \eta \in \Omega_{G(A)}. \quad (2)$$

From the physics point of view, an interacting particle system models a physical setting where basic elements are localized on the vertices of a regular discrete structure, and interact (a priori), only through their nearest neighbors. From a probabilistic point of view, μ_G is a probabilistic model that *factors* through the vertices and edges of the graph. The outstanding feature of such model is the so-called Markov property: We say that the measure μ on Ω_G , satisfies the *global Markov property*, if for all $A \subseteq V$,

$$(\sigma_A : \sigma_{A^c}) \stackrel{d}{=} (\sigma_A : \sigma_{\partial A})$$

The acclaimed Hammersley - Clifford theorem (see, for example, [25]) states that the class of Markov random fields over the graph G is contained in the class of measures factorizable over cliques of G (in our case, the measure is factorizable over cliques of size 1 and 2). This fact shows the generality of the concept of Gibbs' measure from a probabilistic perspective.

A special class of interacting particle systems are the so-called *monotone systems*. Let \leq be a partial order defined on Ω_G . We say that the interacting particle system associated to a given specification is *monotone* if, given $\sigma \sim \mu_G$, it is the case that for all $v \in V$ and $\eta, \eta' \in \Omega_G$ such that $\eta \geq \eta'$,

$$(\sigma(v) : \sigma_{G/\{v\}} = \eta_{G/\{v\}}) \succeq (\sigma(v) : \sigma_{G/\{v\}} = \eta'_{G/\{v\}}). \quad (3)$$

1.1.3 Glauber dynamics

A particular Markov process that arises in the context of interacting particle systems is the so-called *Glauber dynamics*. Given $\eta \in \Omega_G$, $v \in V$, and $a \in X$, let us define

$$\eta^{(v \rightarrow a)}(w) := \begin{cases} \eta(w) & \text{if } w \neq v \\ a & \text{if } w = v \end{cases}. \quad (4)$$

More generally, if $A \subseteq V$, and $\mathbf{a} : A \rightarrow \mathbf{X}$, let

$$\eta^{(A \rightarrow \mathbf{a})}(w) := \begin{cases} \eta(w) & \text{if } w \neq A \\ \mathbf{a}(w) & \text{if } w \in A \end{cases}.$$

The *single-site Glauber dynamics* is the Markov process with state space Ω_G and kernel \mathcal{K} such that

$$\mathcal{K}(\eta, \eta^{(v \rightarrow a)}) := \frac{1}{|V|} \mathbf{P}_{\sigma \sim \mu_G}(\sigma(v) = a : \sigma_{G/\{v\}} = \eta_{G/\{v\}}).$$

More generally, the *k-block Glauber dynamics* is the Markov process defined over Ω_G , whose kernel is

$$\mathcal{K}(\eta, \eta^{(A \rightarrow \mathbf{a})}) := \frac{1}{p(A)} \mathbf{P}_{\sigma \sim \mu_G}(\sigma_A = \mathbf{a} : \sigma_{G/A} = \eta_{G/A}),$$

where $\{p(A)\}_{A \subseteq \Omega_G}$ is the uniform distribution supported over the sets of the form $B_v(k) := \{v' : d(v, v') \leq k\}$.

If the Glauber dynamics is ergodic, then the stationary distribution of the process is, precisely, the corresponding Gibbs' measure over G . Therefore, the Glauber dynamics can be seen as a sampling mechanism to output instances of the Gibbs' measure in the long run. Also, it can be regarded as a natural physical evolution of the system whose equilibrium is dictated by the associated Gibbs' measure.

Suppose that the Markov chain with kernel \mathcal{K} is ergodic respect to the state space Ω . The *mixing time* $\tau_{\text{mix}}(x, \epsilon)$, where $x \in \Omega$ and $\epsilon > 0$, is the time needed for the chain, starting at x , to be ϵ -close (in total variation) to the stationary distribution. More exactly, let $\{X_t\}_{t \geq 0}$ be the Markov process with kernel \mathcal{K} and $X_0 = x$. If μ_t is the distribution of X_t and μ is the stationary distribution of the chain, then:

$$\tau_{\text{mix}}(x, \epsilon) := \min \{t \geq 0 : \text{TV}(\mu_t, \mu) < \epsilon\}.$$

In particular, we define

$$\tau_{\text{mix}}(\epsilon) := \max_{x \in \Omega} \tau_{\text{mix}}(x, \epsilon) \quad \text{and} \quad \tau_{\text{mix}} := \tau_{\text{mix}}(1/2e).$$

A straightforward ‘boosting’ argument (see [65, Chapter 4], for example) implies that, for $\epsilon \in (0, 1)$, $\tau_{\text{mix}}(\epsilon^k) \leq k\tau_{\text{mix}}(\epsilon)$. In particular,

$$\tau_{\text{mix}}(\epsilon) \leq \lceil \ln(1/\epsilon) \rceil \tau_{\text{mix}}. \quad (5)$$

If the interacting particle system is monotone (recall eq. (3)), the mixing time for the Glauber dynamics is determined by the ‘extremal’ versions of the chain. Let $\{X_t\}_{t \geq 0}$ be the single-site Glauber dynamics such that $X_0 = \eta_{\text{max}}$, and let $\{Y_t\}_{t \geq 0}$ be the single-site Glauber dynamics such that $Y_0 = \eta_{\text{min}}$. If $(\bar{X}_t, \bar{Y}_t)_{t \geq 0}$ is the monotone coupling X_t and Y_t ⁴ then, for any $x \in \Omega$,

$$\tau_{\text{mix}}(x, \epsilon)(\epsilon) \leq \min \{t : \mathbb{P}(\bar{X}_t, \bar{Y}_t) < \epsilon\}. \quad (6)$$

Let $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_{|\Omega|}$ be the eigenvalues of the kernel \mathcal{K} . The *spectral gap*, c_{gap} , of the chain is defined as $c_{\text{gap}} := 1 - \max\{\gamma_2, |\gamma_{|\Omega|}|\}$, and the *relaxation time*, τ_{rel} , as $\tau_{\text{rel}} := c_{\text{gap}}^{-1}$, the inverse of the spectral gap. The relaxation time is used as a measure of the convergence rate of a Markov chain, in particular, due to the following relation with the mixing time of the chain (see, e.g., Chapter 12 in [65]):

$$(\tau_{\text{rel}} - 1) \log(1/2\epsilon) \leq \tau_{\text{mix}}(\epsilon) \leq \tau_{\text{rel}} \log(1/\epsilon\mu_{\text{min}}), \quad (7)$$

where $\mu_{\text{min}} := \min \{\mu(\eta) : \eta \in \text{supp } \mu\}$.

The *Dirichlet form* associated with the Markov chain is the (nonlinear) functional $\mathcal{E} : L^2(\Omega) \rightarrow \mathbb{R}$ such that

$$\mathcal{E}(f) := \frac{1}{2} \sum_{\eta, \eta' \in \Omega} [f(\eta) - f(\eta')] \mu(\eta) \mathcal{K}(\eta, \eta'). \quad (8)$$

Let us define

$$\mu(f) := \mathbf{E}_{\sigma \sim \mu} [f(\sigma)] \quad \text{and} \quad \text{Var}_{\mu}(f) := \mathbf{E}_{\sigma \sim \mu} \left[(f(\sigma) - \mu(f))^2 \right].$$

The following variational relation gives an alternative characterization of the gap when the chain is reversible (for instance, for the Glauber dynamics).

$$c_{\text{gap}} = \sup_{\substack{f \in L^2(\Omega) \\ \text{Var}_{\mu}(f) > 0}} \frac{\mathcal{E}(f)}{\text{Var}_{\mu}(f)}. \quad (9)$$

⁴The monotone coupling is a markovian grand coupling that preserves the order. For further details see [65]

The *conductance* of a Markov chain is given by $\Phi := \min_{S \subseteq \Omega} \{\Phi_S\}$, where Φ_S , the conductance of a specific set $S \subseteq \Omega$, is defined as

$$\Phi_S = \frac{\sum_{\eta \in S} \sum_{\eta' \in S^c} \mu(\eta) \mathcal{K}(\eta, \eta')}{\mu(S) \mu(S^c)}. \quad (10)$$

In the above formula, μ is the stationary distribution of the chain.

A general way to find a good upper bound on the conductance is to find a set S such that the probability of “escaping” from S is relatively small. The well-known relationship between conductance and spectral gap established in [62, 99], implies a corresponding upper bound estimate for the gap:

$$c_{\text{gap}} \leq 2\Phi. \quad (11)$$

Given a family \mathcal{F} of interacting particle systems, the *complexity* of the Glauber dynamics in \mathcal{F} is the minimal function $\phi : \mathbb{N} \rightarrow \mathbb{N}$ such that, for every $G \in \mathcal{F}$ with n vertices, the Glauber dynamics over G (defined by its corresponding specification) is bounded by $\phi(n)$.

1.1.4 Some examples

Example 1.1 (Ising model.). *The Ising model ([52]) is a mathematical model of ferromagnetism in statistical mechanics which brought into much of the initial interest in interacting particle systems among physicists. The set of spins is $X = \{-1, 1\}$, and the interaction among spins is either repulsive (among $+$ and $-$ spins) or attractive (among $+$'s or $-$'s spins), fitting into the expression $\phi(x, y) = e^{-\beta xy}$, where β is a parameter called the inverse temperature of the system. Under what physicists call the ‘absence of an external field’, the spins have a neutral self-interaction $\psi(x) = 1$. On the other hand, if an ‘external field’ \mathcal{J} is present, the self-interaction is given by $\psi(x) = e^{x\mathcal{J}}$.*

Example 1.2 (Independent sets model). *The hard-core, or independent sets model, is studied in statistical physics as a model of lattice gas (see, e.g. [103]), and in operations research as a multi-cast model of calls (see [56]). It is also a natural probabilistic and combinatorial problem which corresponds to counting weighted independent sets of the input graph G . The spins in this model are $X = \{0, 1\}$, where 0 corresponds to an unoccupied site and 1 to an occupied site. The interaction is given by $\phi(x, y) = 1 - xy$, restricting*

neighbours to not being simultaneously occupied. The self-interaction is given $\psi(x) = \lambda^x$, being governed by the ‘fugacity’ parameter λ .

Example 1.3 (Coloring model). *The q -coloring model is a combinatorial model that describes the homomorphisms of a given graph into the complete graph K_q . The set of spins (or colors) is $X = \{1, \dots, q\}$. The interaction is defined as $\phi(x, y) = \mathbf{I}(x \neq y)$, while the self-interaction is constant, that is, $\psi(x) = 1$. The support of this model is the set of proper colorings of the graph, that is, assignments such that no two adjacent vertices have the same spin (color). The associated Gibbs’ measure corresponds to the uniform measure over such proper colorings.*

1.1.5 Quantitative properties

Let \mathcal{F} be a family of interacting particle systems. For $G \in \mathcal{F}$, and $v \in G$, the *range- k influence* over the vertex v , is defined as

$$i_G^{(k)}(v) := \max_{x \in X} \max_{\eta \in \Omega_{L_k(v)}} |\mathbf{P}_{\sigma \sim \mu_G}(\sigma(v) = x : \sigma_{L_k(v)} = \eta) - \mathbf{P}_{\sigma \sim \mu_G}(\sigma(v) = x)|.$$

The *range- k influence* in the family \mathcal{F} is defined as

$$i^{(k)} = \max_{G \in \mathcal{F}} \max_{v \in G} i_G^{(k)}(v).$$

If the family consists of rooted trees, instead, we define

$$i_T^{(k)} := \max_{x \in X} \max_{\eta \in \Omega_{L_k}} |\mathbf{P}_{\sigma \sim \mu_T}(\sigma(r) = x : \sigma_{L_k} = \eta_{L_k}) - \mathbf{P}_{\sigma \sim \mu_T}(\sigma(r) = x)| \quad (12)$$

and

$$i^{(k)} = \max_{T \in \mathcal{F}} i_T^{(k)}.$$

This concept measures the ‘maximum’ influence that an assignment over vertices at distance k from v can cause on the spin at v . An alternative notion is that of an ‘average’ influence, which we define next.

The *range- k correlation* or *range- k average influence* over the vertex v , is defined as

$$I_G^{(k)}(v) := \max_{x \in X} \mathbf{E}_{\sigma \sim \mu_G} |\mathbf{P}_{\eta \sim \mu_G}(\eta(v) = x : \eta_{L_k}(v) = \sigma_{L_k}(v)) - \mathbf{P}_{\eta \sim \mu_G}(\eta(v) = x)|. \quad (13)$$

The *range- k correlation* in the family \mathcal{F} is, correspondingly, defined as

$$I^{(k)} = \max_{G \in \mathcal{F}} \max_{v \in G} I_G^{(k)}(v).$$

Similarly, if the family consists of rooted trees, then

$$I_T^{(k)} := \max_{x \in X} \mathbf{E}_{\sigma \sim \mu_T} |\mathbf{P}_{\eta \sim \mu_T}(\eta(r) = x : \eta_{L_k} = \sigma_{L_k}) - \mathbf{P}_{\eta \sim \mu_T}(\eta(r) = x)|$$

and

$$I^{(k)} = \max_{T \in \mathcal{F}} I_T^{(k)}.$$

We say that there is *decay of influence* in the family if $\lim_{k \rightarrow \infty} i^{(k)} = 0$. Also, we say that there is *decay of correlation*, if $\lim_{k \rightarrow \infty} I^{(k)} = 0$. In the context of trees, $I^{(k)}$ is called also the rate of *reconstruction* [81] of the system.

1.1.6 Infinite volume Gibbs' measures

Dobrushin [29] and Lanford, Ruelle [61] introduced a new way to construct probability measures on infinite product probability spaces that does not immediately yield uniqueness. The key-point of their approach is to replace a system of marginals consistent in the sense of Kolmogorov by a system of regular conditional probabilities with respect to the outside of any finite set [63]. In particular, such approach extends to the infinite case the notion of Gibbs' measure defined on the previous section. The particularity of their definition, specifically the possible nonuniqueness, leads to the theory of *phase transitions* in statistical mechanics

For an infinite graph $G = (V, E)$, we assume that every vertex has finite degree (not necessarily bounded). Given a specification $\gamma = (\psi, \phi)$ defined over the *infinite graph* G , we say that the measure μ supported at Ω_G is *compatible* with γ if the following condition holds:

Condition 1.4 (DLR condition). *Given a finite subset of vertices $A \subseteq V$, and given $\eta \in \Omega_{\partial A}$, let μ_A^η be the Gibbs' measure defined over A according to the specification γ and the boundary condition η (as in eq. (2)). Also, let $X(\eta) = \{\eta' \in \Omega_{A^c} : \eta'_{\partial A} = \eta\}$.*

The measure μ satisfies the DLR (Dobrushin-Lanford-Ruelle) condition if, for every finite $A \subseteq V$, every $\eta \in \Omega_{\partial A}$ and every $\eta' \in \Omega_A$,

$$\mathbf{P}_{\sigma \sim \mu}(\sigma_A = \eta' : \sigma_{A^c} \in X(\eta)) = \mathbf{P}_{\sigma \sim \mu_A^\eta}(\sigma = \eta').$$

The main feature of a Gibbs' measure defined on an infinite graph, just as its finite analogue, is the fact that it satisfies the Markov property: For every **finite** A , if $\sigma \sim \mu$, then it is the case that

$$(\sigma_A : \sigma_{A^c}) \stackrel{d}{=} (\sigma_A : \sigma_{\partial A}).$$

We should point out that, in the previous formula, if A is replaced by an infinite set, then the above equivalence is not necessarily true.

The notions of *influence* and *correlation* defined in the previous section can be extended, correspondingly, to the infinite case. However, we must specify the measure compatible with the specification we are interested in.

One of the central problems in the theory of Gibbs' measures is to describe the Gibbs' measures compatible with a given specification. This is known as the DLR problem [42]. Let $\mathcal{G}(\gamma)$ be the set of measures compatible with the specification γ in a given graph. The nonemptiness of $\mathcal{G}(\gamma)$ depends on properties of the specification and the graph, in particular what is called the *quasilocality condition* (see [63, Chapter 3] and [29, 42]). On the other hand, if $\mathcal{G}(\gamma)$ is nonempty, two fundamental questions arise: (i) Does $\mathcal{G}(\gamma)$ consist of a single element? (*the uniqueness problem*), and, in case $|\mathcal{G}(\gamma)| > 1$: (ii) What are the extremal measures contained in $\mathcal{G}(\gamma)$? (*the extremality problem*). The first problem is related with the notion of influence decay defined previously and has been widely studied. Here, we are interested in locating the threshold **uniq**, depending on the parameters of the system, at which the specification switches from bearing one to many compatible Gibbs measures. An easy criteria under which uniqueness holds was established by Dobrushin [29], with corresponding extensions by Dobrushin/Shlosman [32] and several other authors

[18, 50, 75, 109, 113].

The second problem arises in view of the convexity of the set $\mathcal{G}(\gamma)$ (which, furthermore, is a Choquet simplex ⁵). The underlying physical intuition in such a case, tells us that the extremal elements of $\mathcal{G}(\gamma)$ are the ‘natural’ representatives for a probabilistic description of the system. The problem of extremality is related to the concept of decay of correlation previously defined. In fact, a fundamental notion that links both concepts is the fact that extremality of a Gibbs’ measure is equivalent to the triviality of its tail σ -field (see, e.g., [42]).

Above, we have synthesized most of the definitions necessary for the subsequent exposition. We direct the reader, also, to the extensive literature regarding Gibbs’ measure. For example, [18, 42, 49, 58]. Also, for additional information about standard techniques employed in Markov chain mixing time, we refer the reader to [5, 65, 79].

1.2 Results

This thesis is based on the research articles [78, 97, 96]. Even though, additional material has been introduced and significant changes in the presentation have been made. We summarize next our main findings.

1.2.1 Chapter 2: Tree Geometries.

In this chapter, we study the single-site Glauber dynamics on interacting particle systems defined over regular trees. We focus, in particular, on the independent sets model on regular trees. Our interest in this problem lies on a question that goes back to the work of Martinelli, Sinclair and Weitz [75]. In their work, they left open the problem of the existence of a sequence of boundary conditions that slows down the dynamics. We answer such a question affirmatively. More importantly, we establish a phase transition in the relaxation time of the dynamics, for general boundary conditions.

This work is based on our research article [97]. Our result is based on a general connection between reconstruction ‘schemes’ and the mixing of the Glauber dynamics (Theorem

⁵See [42] for this fact and, for instance, [87] for an introduction to Choquet’s theory.

2.4). We apply this relation to the independent sets model with fugacity $\lambda = \omega(1 + \omega)^b$ in the regular b -ary tree of height h , T_h , to obtain the following:

Theorem. *For all $h \geq 1$, the single site Glauber dynamics for the independent sets model on the tree T_h with boundary condition Γ_h , satisfies (n is the number of vertices of T_h):*

1. *For $\omega \leq \ln b/b$ and arbitrary Γ_h ,*

$$\tau_{rel} \leq C_{\delta,b} n^{1+\epsilon(b)},$$

where $\epsilon(b) \leq c \ln \ln b / \ln b$.

2. *For all $\delta > 0$, $\omega = (1 + \delta) \ln b/b$ and arbitrary Γ_h ,*

$$\tau_{rel} \leq C_{\delta,b} n^{1+\delta+\epsilon(b)}.$$

where $\epsilon(b) \leq c \ln \ln b / \ln b$.

In the following, I is a fixed interval $I = (0, R)$, where R is a fixed arbitrary constant

3. *For all $\delta \in I$, if $\omega = (1 + \delta) \ln b/b$ and Γ_h is the boundary such that $\psi_v^{(\Gamma_h)}(0) = \frac{1}{1+\omega}$ and $\psi_v^{(\Gamma_h)}(1) = \frac{\omega}{1+\omega}$,*

$$\tau_{rel} \geq C_{\delta,b} n^{1+\delta/2+\epsilon(b)},$$

where $\epsilon(b) \leq c \ln \ln b / \ln b$.

4. *For all $\delta \in I$, $\omega = (1 + \delta) \ln b/b$, there exists a sequence of discrete boundary conditions $\{\Gamma_h\}_{h \geq 1}$, such that*

$$\tau_{rel} \geq C_{\delta,b} n^{1+\delta/2+\epsilon(b)},$$

where $\epsilon(b) \leq c \ln \ln b / \ln b$.

1.2.2 Chapter 3: Lattice geometries.

In this chapter, we focus on the well-studied case of the square lattice \mathbb{Z}^2 and provide a new lower bound for the uniqueness threshold of the independent sets model. Our technique refines and builds on the tree of self-avoiding walks approach of Weitz [114], resulting in a new technical sufficient criterion (of wider applicability) for establishing strong spatial

mixing (and hence uniqueness) for the independent sets model. Our new criterion achieves better bounds on strong spatial mixing when the graph has extra structure, improving upon what can be achieved by just using the maximum degree. Applying our technique to \mathbb{Z}^2 , we prove that strong spatial mixing (a measure of influence between far away vertices under arbitrary ‘perturbations’ of the system, see Section 3.2) holds for all $\lambda < 2.3882$, improving upon the work of Weitz that held for $\lambda < 27/16 = 1.6875$. Our results imply a fully-polynomial deterministic approximation algorithm for estimating the partition function, as well as rapid mixing of the associated Glauber dynamics to sample from the independent sets distribution for $\lambda < 2.3882$.

The results in this chapter are mainly based on our research article [96]. The following is our main result (Corollary 3.21):

Theorem. *The following holds for the independent sets model with fugacity λ in \mathbb{Z}^2 , for all $\lambda \leq \lambda^* = 2.3882$.*

1. *SSM holds for finite subgraphs of \mathbb{Z}^2 . That is, there exists $C > 0$ and $\gamma \in (0, 1)$ such that for every finite subgraph G of \mathbb{Z}^2 , every $v \in G$ and every $k \geq 1$,*

$$ssm_{G,v}^{(k)} \leq C\gamma^k.$$

2. *There exists an unique infinite-volume Gibbs’ measure for the independent sets model in \mathbb{Z}^2 .*
3. *For any $\lambda \leq \lambda^*$, there exist constants $C, c > 0$, such that Weitz’ algorithm [114] calculates an ϵ -approximation of the partition function $Z(G)$ for any $G \subseteq \mathbb{Z}^2$, in time $C(n/\epsilon)^c$, where $n = |G|$.*
4. *For any $\lambda \leq \lambda^*$, there exists $C > 0$, such that the Glauber dynamics mixes in $Cn \log n$ iterations, for any finite subgraph G of \mathbb{Z}^2 , where $n = |G|$.*

Further, we establish a criterion (Proposition ??) for general spin systems, that captures the role of the connectivity of the graph in the spatial mixing of the system. In particular, for the well studied Ising model, such a criterion fits the actual tight estimates known for general trees.

1.2.3 Chapter 4: Random geometries.

In this chapter, we study the decay of correlation for the q -coloring model on sparse random graphs. This ‘correlation’ problem has been studied in detail in the context of Gibbs measures on trees [15, 81, 101]. For random graphs, as we do here, Montanari and Gerschenfeld [43] initiated a program, in which under a ‘sphericity’ condition, they obtain a direct translation of correlation rates between the random graph and a corresponding Poisson random tree. We take such direction for our study.

Our interest in this model lies in the fact that random structures provide an outstanding example of phase transitions in statistical physics and combinatorics. They render one of the ‘natural’ situations in which an underlying algorithmic ‘hardness’ appears when the density of the random graph increases. For instance, it is well-known that it is easy to color a random graph using twice as many colors as its chromatic number, by using a simple greedy strategy. On the other hand, yet to date, no algorithm is known that uses less than twice the chromatic number of the graph. This factor of 2 corresponds in a precise mathematical sense to a phase transition in the geometry of the support of the Gibbs’ measure for the q -coloring model, called *clustering* [1]. Our aim in this chapter is to determine the existence of correlations between distant vertices, in the same regime, providing further evidence for this conjectural hardness.

This chapter is based on our research article [78]. The following is the main result of the chapter (Theorem 4.1):

Theorem. *Let us consider the (random) Gibbs’ measure for the q -coloring specification in the random graph $G(n, \alpha)$ (Erdős-Rényi random graph with average degree α).*

1. *For every $\delta > 0$, there exists $q_0(\delta)$ such that for all $q > q_0(\delta)$, if $\alpha = (1 + \delta) q \log q$, the q -coloring model in $G(n, \alpha)$ exhibits correlation (see page 84 for the exact definition).*
2. *For every $\delta > 0$, there exists $q_0(\delta)$ such that for all $q > q_0(\delta)$, if $\alpha = (1 - \delta) q \log q$, the q -coloring model in $G(n, \alpha)$ has vanishing correlation.*

CHAPTER II

TREE GEOMETRIES

In this chapter, we consider interacting particle systems defined over regular trees. In particular, it is our objective to describe a phase transition behaviour for the single-site Glauber dynamics for the independent sets model on regular trees. Our interest in such a problem lies on a challenging question that goes back to the work of Martinelli, Sinclair and Weitz [75]. In their work, they left open the question of the existence of a sequence of boundary conditions that slows down the dynamics. We will answer such a question affirmatively, furthermore, we establish a phase transition in the relaxation time of the dynamics for general boundary conditions.

The study of finite tree geometries for interacting particle systems is regarded as a canonical ‘easy’ example due to the recursive formulation of the Gibbs’ measures associated to a given specification. In the case of infinite trees, the situation gets more complex, but the availability of finite approximations (when possible) makes the study tractable. The importance of tree-systems lies also in the fact that several results carry out to the case of more general graphs. For instance, we will see in Chapter 3 how uniqueness on trees is related to rapid mixing of the Glauber dynamics on lattices and, in Chapter 4 we will relate extremality on trees with finite decay of correlations on sparse random graphs. Some other references that exhibit the crucial role of tree geometries in the study of general interacting particle systems are [1, 43, 114].

This chapter is divided into three sections. In Section 2.1, we present general properties and notation specific for Gibbs’ measures defined over tree geometries. In Section 2.2, we define the concept of a reconstruction scheme, exhibiting a quantitative relation between the notion of sensitivity of a scheme and the relaxation time of the dynamics. In Section 2.3, we prove our main theorem, which establishes a phase transition in the mixing time for the Glauber dynamics in the independent sets model on trees.

2.1 Gibbs' measures over trees

The (regular) b -ary tree, T , is a connected acyclic graph, such that every vertex has degree $b + 1$, except for a distinguished vertex r called the root of the graph, which has degree b . In this case, we say that the tree has *branching* b . We will denote by T_h the finite tree that consists of the restriction of T to the vertices at distance $\leq h$ from the root. The *leaves* of the tree T_h are the vertices at distance h from the root. More generally, the level L_k of the tree, is the set of vertices at distance k from the root. Given a vertex $v \in T$ such that $d(v, r) = l$, we say that the vertices v_1, \dots, v_b are the children of v , if for every $i = 1, \dots, b$, $v_i \sim v$ and $d(v_i, r) = l + 1$. The *tree subtended at v* , $T_h^{(v)}$, is the subtree induced by the vertices u such that the path joining u and r contains the vertex v .

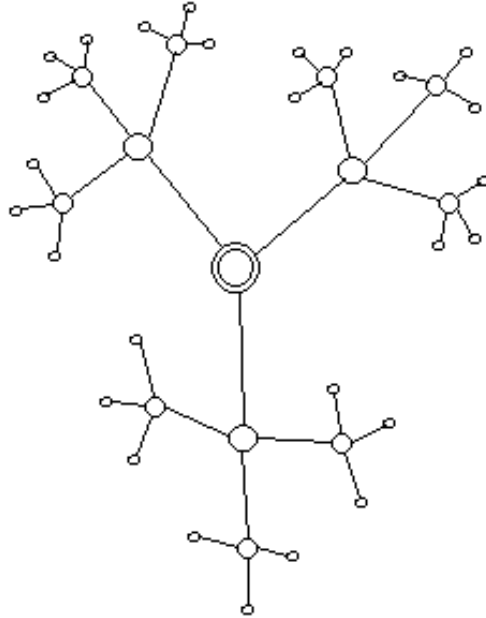


Figure 1: Regular tree of branching 3.

Given a specification γ over the infinite tree T , in order to define a Gibbs' measure over the finite tree T_h we (may) additionally specify a *boundary condition* Γ_h . This boundary condition is a redefinition $\psi_v^{(\Gamma_h)}$ of the self-interactions of the specification, at the leaves of the tree. Some types of boundary conditions are:

- *Free boundary condition.* The fugacity at the leaves is the same, that is

$$\psi_v^{(\Gamma_h)} := \psi_v.$$

- *Discrete boundary condition.* The fugacity is redefined in such a way that is concentrated at a given spin. That is, for some $\{a_v\}_{v \in L_h}$, where $a_v \in X$,

$$\psi_v^{(\Gamma_h)}(x) := \mathbf{I}(x = a_v), \quad x \in X.$$

- *Uniform boundary condition.* The fugacity is redefined uniformly among the leaves, that is

$$\psi_v^{(\Gamma_h)} = \psi_w^{(\Gamma_h)} \text{ for all } v, w \in L_h.$$

Given a boundary Γ_h , we denote by $\mu_{T_h, \Gamma_h}(\cdot)$, or in short notation $\mu_h(\cdot)$, the Gibbs' measure associated with the specification γ , with boundary condition Γ_h , over the tree T_h . Given $v \in T_h$, we denote by $\Gamma_h^{(v)}$ the boundary condition restricted to the subtree subtended at v . We denote by $\mu_{T_h^{(v)}, \Gamma_h^{(v)}}(\cdot)$, or in short notation $\mu_{h,v}(\cdot)$, the Gibbs' measure compatible with the specification γ , in the tree $T_h^{(v)}$, with boundary condition $\Gamma_h^{(v)}$.

The following recurrence holds for Gibbs' measures over finite trees. For every $\eta \in \Omega_{T_h}$, if v_1, \dots, v_b are the children of the root r , then it is the case that

$$\mathbf{P}_{\sigma \sim \mu_h}(\sigma = \eta) = \frac{\psi(\eta(r)) \prod_{i=1}^b \left[\phi(\eta(r), y) \mathbf{P}_{\sigma \sim \mu_{h,v_i}}(\sigma = \eta_{T_h^{(v_i)}}) \right]}{\sum_{x' \in X} \psi_r(x') \prod_{i=1}^b \left(\sum_{y \in X} \phi_{(r,v_i)}(x', y) \mathbf{P}_{\sigma \sim \mu_{h,v_i}}(\sigma(v_i) = y) \right)} \quad (14)$$

and, in particular,

$$\mathbf{P}_{\sigma \sim \mu_h}(\sigma(r) = x) = \frac{\psi_r(x) \prod_{i=1}^b \left(\sum_{y \in X} \phi_{(r,v_i)}(x, y) \mathbf{P}_{\sigma \sim \mu_{h,v_i}}(\sigma(v_i) = y) \right)}{\sum_{x' \in X} \psi_r(x') \prod_{i=1}^b \left(\sum_{y \in X} \phi_{(r,v_i)}(x', y) \mathbf{P}_{\sigma \sim \mu_{h,v_i}}(\sigma(v_i) = y) \right)}. \quad (15)$$

This recurrence does not necessarily hold for Gibbs' measures over an infinite tree, but a similar recurrence in terms of *broadcastings* or *Markov processes over edges* does hold in the case that the measure is *simple*, as we will introduce next.

Given a vertex $v \in T$, let $T^{(1)}, \dots, T^{(b+1)}$ be the connected components of $T/\{v\}$ (if v is the root of T , we obtain b connected components instead). A Gibbs' measure μ over T is *simple* [20], if for every vertex v ,

$$\mathbf{P}_{\sigma \sim \mu} (\sigma_{T^{(i)}} = \eta_{T^{(i)}} \text{ for } i = 1, \dots, b+1 : \sigma(v) = \eta(v)) = \prod_{i=1}^{b+1} \mathbf{P}_{\sigma \sim \mu} (\sigma_{T^{(i)}} = \eta_{T^{(i)}} : \sigma(v) = \eta(v)).$$

That is, for $\sigma \sim \mu$, $\sigma_{T^{(1)}}, \dots, \sigma_{T^{(b+1)}}$ are independent **given** $\sigma(v)$. Notice that if T is a finite tree, the measure is obviously simple, while if the tree is infinite, this property does not necessarily hold ¹.

If the measure μ is simple, then, there exists a probability measure π supported on X and, for every edge $e \in E(T)$, there exists a stochastic matrix m_e , with state space X , such that the following holds.

- If the tree is finite, then for every $\eta \in \Omega_T$

$$\mathbf{P}_{\sigma \sim \mu} (\sigma = \eta) = \pi(\eta(r)) \prod_{(v,w) \in E(T)} m_{(v,w)}(\eta(v), \eta(w))$$

- If the tree is infinite, then, for every $h \geq 1$ and every $\eta \in \Omega_{T_h}$ (where T_h is the restriction of the tree to the first h levels),

$$\mathbf{P}_{\sigma \sim \mu} (\sigma_{T_h} = \eta) = \pi(\eta(r)) \prod_{(v,w) \in E(T_h)} m_{(v,w)}(\eta(v), \eta(w))$$

A measure of the above form is called a *broadcasting process* [81]. Notice that, in the case of a finite tree, we have, in view of Eq. (14) and (15), that for every edge (v, w) in T_h where w is a child of v ,

$$m_{(v,w)}(x, y) = \frac{\phi_{(v,w)}(x, y) \mathbf{P}_{\sigma \sim \mu_{h,w}}(\sigma(w) = y)}{\sum_{z \in X} \phi_{(v,w)}(x, z) \mathbf{P}_{\sigma \sim \mu_{h,w}}(\sigma(w) = z)}. \quad (16)$$

In particular, if such ‘broadcastings’ m_e are uniform among the edges of the tree, that is, if for all edges e of T , $m_e = m$ for a fixed stochastic matrix m , we say that the simple Gibbs' measure is *invariant*. It is a standard fact that every uniform specification γ over a b -ary infinite tree T has at least one *simple invariant version* (see, e.g., [20]). Moreover,

¹For instance, the measure $\frac{1}{2}\nu^+ + \frac{1}{2}\nu^-$, where ν^+ and ν^- are defined in example 2.1.

such a simple invariant version can be realized on finite trees by choosing an appropriate boundary condition. In fact, if $\pi \in [0, 1]^X$ is a fixed point of the system of equations

$$\pi(x) = \frac{\psi(x) \left(\sum_{z \in X} \phi(x, z) \pi(z) \right)^b}{\sum_{x' \in X} \psi(x') \left(\sum_{z \in X} \phi(x', z) \pi(z) \right)^b}, \quad x \in X$$

and

$$m(x, y) = \frac{\phi(x, y) \pi(y)}{\sum_{z \in X} \phi(x, z) \pi(z)}, \quad x, y \in X$$

then, it is easy to see that, if ν is the broadcasting measure over T , induced by π and m , then:

- ν is a simple invariant Gibbs' measure compatible with the specification γ .
- Let ν_h be the measure ν restricted to the tree T_h , and let μ_h be the Gibbs' measure over T_h , compatible with the specification γ and with boundary condition Γ_h such that

$$\psi_v^{(\Gamma_h)}(x) := \pi(x), \quad v \in L_h, x \in X.$$

Then, $\nu_h = \mu_h$.

Notice that the existence of π is guaranteed by the Brouwer's fixed point theorem. However, the question of uniqueness is more subtle. We refer the reader to the monographs [20] and [104] for this and other related questions.

Further extensions to the concept of invariance can be defined. For instance, a *2-periodic simple* version of a specification, is a Gibbs' measure generated by broadcastings (therefore, simple), and such that $m_e = m_{\text{even}}$ if $d(r, e)$ is even and $m_e = m_{\text{odd}}$ if $d(r, e)$ is odd, where $m_{\text{even}}, m_{\text{odd}}$ are a pair of fixed stochastic matrices with state space X .

Several monographs about Gibbs' measures over trees, which, by itself, is a topic of considerable interest are [22, 20, 42, 104]. We refer the interested reader to these, for additional information.

Example 2.1 (The independent sets model.). *For the independent sets model (defined in Example 1.2) on the b -ary tree, there exists a unique simple invariant measure that we*

denote by $\nu(\cdot)$. This measure is defined by the broadcasting process such that $\pi(0) = \frac{1}{1+\omega}$, $\pi(1) = \frac{\omega}{1+\omega}$, and $m = \begin{bmatrix} \frac{1}{1+\omega} & \frac{\omega}{1+\omega} \\ 1 & 0 \end{bmatrix}$, where ω is the unique solution to the equation $\omega(1+\omega)^b = \lambda$. The restriction of ν to the finite tree T_h can alternatively be defined as the Gibbs' measure compatible with the independent sets specification on T_h , with boundary condition Γ_h such that $\psi_v^{(\Gamma_h)}(0) = \frac{1}{1+\omega}$ and $\psi_v^{(\Gamma_h)}(1) = \frac{\omega}{1+\omega}$. We will denote such restriction by $\nu_h(\cdot)$. It will be customary for us to take ω as the actual parameter for the independent sets measure instead of λ , having in mind always the relation $\lambda = \omega(1+\omega)^b$.

If it is the case that $\omega \leq \frac{1}{b-1}$, the measure described above is the unique Gibbs' measure on the infinite tree T compatible with the specification and, therefore, every finite approximation will converge to it. If $\omega > \frac{1}{b-1}$, then at least 3 extremal measures exist: ν and two 2-periodic measures that we denote by ν^+ and ν^- . The measure ν^+ , for instance, is the limiting measure resulting from the finite approximation over the trees T_h with boundary condition Γ_h that assigns the spin 0 to the leaves if h is even, and the spin 1 if h is odd. Equivalently, ν^+ is the measure generated by the broadcasting process such that $\pi(0) = \alpha^+$, $\pi(1) = 1 - \alpha^+$, $m_{\text{even}} = \begin{bmatrix} \alpha^- & 1 - \alpha^- \\ 1 & 0 \end{bmatrix}$, $m_{\text{odd}} = \begin{bmatrix} \alpha^+ & 1 - \alpha^+ \\ 1 & 0 \end{bmatrix}$, where α^+ and α^- are, respectively, the greatest and smallest fixed points of the function $f(x) = \frac{1}{1+\lambda\left(\frac{1}{1+\lambda x^b}\right)^b}$. A similar description holds for ν^- , by switching the words even and odd in the previous two sentences.

If $\omega = \frac{(1+\delta)\log b}{b}$ (for b large enough), the measure ν is not extremal. Moreover, ν is not a convex combination of ν^+ and ν^- , therefore, implying a richer structure in the simplex generated by the Gibbs' measures compatible with the independent sets specification over T .

For the question of uniqueness of this model we refer the reader to [56, 96, 114], and regarding extremality, some references are [14, 23, 72, 75].

2.2 Non-extremality and mixing

Extremality of a simple Gibbs' measure μ on an infinite tree T is related to the so-called reconstruction problem on trees. Such a problem asks if for $\sigma \sim \mu$, there is a non-vanishing correlation between $\sigma(r)$ and σ_{L_h} as $h \rightarrow \infty$ that allows to recover ancestral information

$(\sigma(r))$ from present information (σ_{L_h}) with nontrivial success. If such is the case, we say that there is *reconstruction* for the model. This is a question inspired in several types of statistical analysis and is especially important in phylogenetics [27].

More precisely, the reconstruction problem asks if there is (or not) decay of correlation with respect to the distinguished root of the tree ². Therefore, there is reconstruction if and only if the measure is non-extremal. The value of the parameter of the model (if it exists), where a transition between reconstruction and non-reconstruction takes place is called the *reconstruction threshold* of the model and is denoted by the subscript ‘**rec**’.

Conditions implying non-reconstruction are established in [75] and [60]. For tight analysis of the reconstruction regime for several models, we refer the reader to [15, 14, 101]. Also, for a survey regarding reconstruction on trees, see [81].

A general connection between reconstruction (i.e. nonextremality) and the mixing time of the Glauber dynamics was shown by Berger, Kenyon, Mossel and Peres [13], who proved, for general spin systems, that $O(n)$ relaxation time implies nonreconstruction. On the other hand, in several situations it has been observed that extremality (i.e. nonreconstruction) implies $O(n \log n)$ mixing time, suggesting a phase transition of the Glauber dynamics when the corresponding phase transition for extremality occurs. This is a conjecture (or rather, a question) that was raised by the authors in [13]. This conjecture is supported by several works in which such a phase transition has been exhibited [13, 28, 45, 69, 108]. Furthermore, this connection has also been observed in sparse random graphs [1] and planar graphs [51].

Our goal next, is to understand the relationship between extremality of the Gibbs’ measure and mixing time of the Glauber dynamics, by transforming an iterative ‘scheme’ that shows reconstruction, into a set with poor conductance, which implies a lower bound in the relaxation time (an approach used also in [108]). In particular, we exhibit a quantitative relation between the ‘sensitivity’ of a reconstruction scheme and the relaxation time of the dynamics. In Section 2.3, we employ such an analysis to show a phase transition in the mixing time of Glauber dynamics for the independent sets model on trees.

²Recall the definition of correlation decay from page 11.

2.2.1 Reconstruction schemes and sensitivity analysis

For a fixed specification γ on an infinite tree T , and a sequence of boundary conditions $\{\Gamma_h\}_{h \geq 1}$, let μ_h be the Gibbs' measure associated with the specification γ over T_h , with boundary condition Γ_h . Given a fixed spin $a \in X$, we say that the sequence of functions $F_{h,k} : \Omega_{T_h} \rightarrow \{0, 1\}$, where $h \geq 1$ and $k = k(h)$, is a *reconstruction scheme* for guessing the spin a at the root if:

1. $F_{h,k}(\eta)$ depends only on η_{L_k} .³
2. The events $\{\eta : F_{h,k}(\eta) = 1\}$ and $\{\eta : \eta(r) = a\}$ are μ_h -positively correlated.

Essentially, the function F takes the configuration at level k as the input (forgetting the remaining information) and tries to 'guess' if the spin a was (or not) assigned at the root, in the original configuration.

The *effectiveness* of F is the following measure of the covariance between F 's output and the actual marginal at the root.

$$\text{eff}_h(F) := \mathbf{P}_{\sigma \sim \mu_h}(F_{h,k}(\sigma) = 1 \text{ and } \sigma(r) = a) - \mathbf{P}_{\sigma \sim \mu_h}(F_{h,k}(\sigma) = 1) \mathbf{P}_{\sigma \sim \mu_h}(\sigma(r) = a).$$

We say that F is *nontrivial* if the event $\{\eta : F_{h,k}(\eta) = 1\}$ is nontrivial, that is

$$\liminf_{h \rightarrow \infty} \mu_h(\{\eta : F_{h,k}(\eta) = 1\}) (1 - \mu_h(\{\eta : F_{h,k}(\eta) = 1\})) > 0$$

Similarly, for a measure μ compatible with the specification γ on the **infinite tree** T , a reconstruction scheme for the spin $a \in X$ is a sequence of functions $F_h : \Omega_T \rightarrow \{0, 1\}$ for $h \geq 1$, such that

1. $F_h(\eta)$ depends only on η_{L_h} .
2. The events $\{\eta : F_h(\eta) = 1\}$ and $\{\eta : \eta(r) = a\}$ are μ -positively correlated.

The *effectiveness* of F in this case is defined as follows.

$$\text{eff}_h(F) := \mathbf{P}_{\sigma \sim \mu}(F_h(\sigma) = 1 \text{ and } \sigma(r) = a) - \mathbf{P}_{\sigma \sim \mu}(F_h(\sigma) = 1) \mathbf{P}_{\sigma \sim \mu}(\sigma(r) = a).$$

³Recall that L_k is the set of vertices at distance k from the root.

Also, we say that the reconstruction scheme F is *nontrivial* if the event $\{\eta : F_h(\eta) = 1\}$ is nontrivial, that is

$$\liminf_{h \rightarrow \infty} \mu(\{\eta : F_h(\eta) = 1\}) (1 - \mu(\{\eta : F_h(\eta) = 1\})) > 0$$

In the infinite case, the non-triviality of F implies the non-triviality of the tail event $\bigcap_{h' \geq h} \{\eta : F_{h'}(\eta) = 1\}$ and, therefore, the non-extremality of the Gibbs' measure (see [42]). For the finite case, the situation is more subtle: we require certain appropriate behaviour of the level index $k = k(h)$ to guarantee the non-extremality of the limiting Gibbs' measure. Nevertheless, we can establish the following relation between the effectiveness of a reconstruction scheme and the decay of correlation with respect to the root:

Proposition 2.2. *Let $F_{h,k}$ be a reconstruction scheme. Then, it is the case that ⁴*

$$\text{eff}_h(F) \leq I_{T_h}^{(k)}.$$

Proof. Simply notice that

$$\begin{aligned} I_{T_h}^{(k)} &= \max_{x \in X} \mathbb{E}_{\eta \sim \mu_h} |\mathbf{P}_{\sigma \sim \mu_h}(\sigma(r) = x : \sigma_{L_k} = \eta_{L_k}) - \mathbf{P}_{\sigma \sim \mu_h}(\sigma(r) = x)| \\ &\geq \mathbb{E}_{\eta \sim \mu_h} [(\mathbf{P}_{\sigma \sim \mu_h}(\sigma(r) = a : \sigma_{L_k} = \eta_{L_k}) - \mathbf{P}_{\sigma \sim \mu_h}(\sigma(r) = a)) \mathbf{I}(F_{h,k}(\eta) = 1)] \\ &= \text{eff}_h(F). \end{aligned}$$

□

Definition 2.3. *The sensitivity of the reconstruction scheme $F_{h,k}$, at the configuration $\eta \in \Omega_{T_h}$, is the fraction of vertices $v \in L_k$ such that switching the spin at v in η changes the final output of $F_{h,k}$ from 1 to 0. More precisely, recalling that $\eta^{(v=x)}$ is the configuration that differs from η only at the spin v ⁵, we define the sensitivity as:*

$$\nabla F_{h,k}(\eta) = \begin{cases} \frac{1}{|L_k|} \# \{v \in L_k : \text{for some } x \in X, F_{h,k}(\eta^{(v=x)}) = 0\} & \text{if } F_{h,k}(\eta) = 1 \\ 0 & \text{if } F_{h,k}(\eta) = 0 \end{cases}$$

⁴Recall, from page 13, the definition of the range- k correlation $I_{T_h}^{(k)}$.

⁵See, Eq. (4), page 6.

The following theorem establishes a link between the average sensitivity of a nontrivial reconstruction scheme and the relaxation time of the single-site Glauber dynamics.

Theorem 2.4. *Suppose that $F_{h,k}$ is a **nontrivial** reconstruction scheme for the sequence of Gibbs' measures μ_h over the tree T_h , with boundary condition Γ_h .*

Then, there exists a constant C , such that for all $h \geq 1$, the relaxation time τ_{rel} of the Glauber dynamics for μ_h satisfies

$$\tau_{\text{rel}} \geq \frac{Cn^{(1-k/h)}}{\mathbf{E}_{\sigma \sim \mu_h} [\nabla F_{h,k}(\sigma)]}, \quad (17)$$

where $n = |V(T_h)| = \frac{b^{h+1}-1}{b-1}$.

Proof. The conductance⁶ of the set $U_k = \{\eta : F_{h,k}(\eta) = 1\}$ is given by

$$\begin{aligned} \Phi_{U_k} &= \frac{\sum_{\sigma} \mu_h(\sigma) \sum_{v \in L_k} \sum_{x \in X} \mathcal{K}(\sigma, \sigma^{(v \rightarrow x)}) \mathbf{I}(F_{h,k}(\sigma^{(v \rightarrow x)}) = 0)}{\mu_h(U_k)(1 - \mu_h(U_k))} \\ &\leq \frac{C'}{n^{(1-k/h)}} \frac{\sum_{\sigma \in U} \mu_h(\sigma) \nabla F_{h,k}(\sigma)}{\mu_h(U_k)(1 - \mu_h(U_k))}, \\ &= \frac{C''}{n^{(1-k/h)}} \mathbf{E}_{\sigma \sim \mu_h} [\nabla F_{h,k}(\sigma)], \text{ due to the non-triviality of } F_{h,k}, \end{aligned}$$

where the constant C'' does not depend on h . Therefore, due to Eq. (11) (page 9), we have

$$\tau_{\text{rel}} = (c_{\text{gap}})^{-1} \geq \frac{1}{2\Phi_{U_k}} \geq \frac{Cn^{(1-k/h)}}{\mathbf{E}_{\sigma \sim \mu_h} [\nabla F_{h,k}(\sigma)]},$$

where $C = 1/2C''$. □

The m.l.e (maximum likelihood estimator) reconstruction scheme is defined, in the finite tree case, as follows:

$$F_{h,k}(\sigma) := 1 \text{ iff } \mathbf{P}_{\sigma \sim \mu_h}(\sigma_{L_k} : \sigma(r) = a) > \mathbf{P}_{\sigma \sim \mu_h}(\sigma_{L_k} : \sigma(r) = x) \text{ for all } x \neq a.$$

A similar definition is employed in the infinite tree case.

In the finite tree case, the effectiveness of the m.l.e. scheme, for some spin a , is at least $\frac{1}{2}I_{T_h}^{(k)}(r)$. Moreover, for infinite trees, it is easy to prove that the measure is non-extremal iff for some spin a the m.l.e. reconstruction scheme is nontrivial. Thus, this reconstruction

⁶As defined in Eq. (10) in page (10)

scheme is adequate to be used in Theorem 2.4, all over the non-reconstruction region. Unfortunately, the m.l.e. scheme is quite complicated to analyze in general, and we must appeal to easier schemes of reconstruction. Next we introduce an easy, appropriate one for the independent sets model.

2.2.2 A parsimonious scheme

Based on an algorithm of Brightwell and Winkler [23] (see also [81]), which can be regarded as a “short memory” m.l.e. reconstruction scheme for the independent sets model, we define the following reconstruction scheme. This will be useful to analyze the relation between extremality and relaxation time for the independent sets model. We will refer to it as the *parsimonious scheme* later on; see section 2.2.2.1.

First, we define the auxiliary function $G_{h,k}(\eta, v)$ for $\eta \in \Omega_{T_h}$ and $v \in T_h$, in the following way:

1. If $v \in L_k$, then

$$G_{h,k}(\eta, v) = \begin{cases} 1 & \text{if } \eta(v) = a \\ 0 & \text{otherwise} \end{cases}.$$

2. Inductively, for $v \notin L_k$, if w_1, \dots, w_b are the children of v , then

$$G_{h,k}(\eta, v) = \prod_{i=1}^b (1 - G_{h,k}(\eta, w_i)).$$

Finally, we define $F_{h,k}(\eta) := G_{h,k}(\eta, r)$, where r is the root of T_h .

It is clear that $F_{h,k}(\eta)$ depends only on η_{L_k} . On the other hand, to show the positive correlation of the events $\{\eta : F_{h,k}(\eta) = 1\}$ and $\{\eta : \eta(r) = a\}$, we need additional conditions. In the case of 2-spin systems, a sufficient condition is given by $m_e(0, 1) \geq m_e(1, 1)$, for all edges e in T_h . This is proved in the following proposition.

Proposition 2.5. *For the reconstruction scheme $F_{h,k}(\sigma)$ defined above, the events $\{\eta : F_{h,k}(\eta) = 1\}$ and $\{\eta : \eta(r) = 1\}$ are positively correlated if $m_e(0, 1) \geq m_e(1, 1)$ for every edge e in T_h .*

Proof. For $v \in T_h$, define

$$\alpha_v := \mathbf{P}_{\sigma \sim \mu_h}(G_{h,k}(\sigma, v) = 1 : \sigma(v) = 1)$$

and

$$\beta_v := \mathbf{P}_{\sigma \sim \mu_h} (G_{h,k}(\sigma, v) = 1 : \sigma(v) = 0).$$

In particular, $\alpha_v = 1$ and $\beta_v = 0$ for $v \in L_k$. Now, for any $v \in T_h$ with children w_1, \dots, w_b , the following recurrence takes place:

$$\begin{aligned} \alpha_v &= \mathbf{P}_{\sigma \sim \mu_h} (G_{h,k}(\sigma, v) = 1 : \sigma(v) = 1) \\ &= \mathbf{P}_{\sigma \sim \mu_h} \left(\prod_{i=1}^b (1 - G_{h,k}(\sigma, w_i)) = 1 : \sigma(v) = 1 \right), \text{ by definition of } G_{h,k} \\ &= \prod_{i=1}^b \mathbf{P}_{\sigma \sim \mu_h} (G_{h,k}(\sigma, w_i) = 0 : \sigma(v) = 1), \text{ due to the simplicity of the measure} \\ &= \prod_{i=1}^b [m_{(v, w_i)}(1, 1)(1 - \alpha_{w_i}) + m_{(v, w_i)}(1, 0)(1 - \beta_{w_i})] \end{aligned} \tag{18}$$

and similarly,

$$\begin{aligned} \beta_v &= \mathbf{P}_{\sigma \sim \mu_h} (G_{h,k}(\sigma, v) = 1 : \sigma(v) = 0) \\ &= \prod_{i=1}^b [m_{(v, w_i)}(0, 1)(1 - \alpha_{w_i}) + m_{(v, w_i)}(0, 0)(1 - \beta_{w_i})]. \end{aligned} \tag{19}$$

Now, assuming that $\alpha_{w_i} \geq \beta_{w_i}$ for every i , we have that

$$\begin{aligned} &[m_{(v, w_i)}(1, 1)(1 - \alpha_{w_i}) + m_{(v, w_i)}(1, 0)(1 - \beta_{w_i})] \\ &- [m_{(v, w_i)}(0, 1)(1 - \alpha_{w_i}) + m_{(v, w_i)}(0, 0)(1 - \beta_{w_i})] \\ &= (m_{(v, w_i)}(0, 1) - m_{(v, w_i)}(1, 1))(\alpha_{w_i} - \beta_{w_i}) \\ &\geq 0, \end{aligned}$$

thus, implying that $\alpha_v \geq \beta_v$. Therefore, by induction, we get that for every vertex v in the tree, $\alpha_v \geq \beta_v$. In particular, for the root, $\alpha_r \geq \beta_r$. Therefore,

$$\begin{aligned} &\mathbf{P}_{\sigma \sim \mu_h} (F_{h,k}(\sigma) = 1 : \sigma(r) = 1) - \mathbf{P}_{\sigma \sim \mu_h} (F_{h,k}(\sigma) = 1) \\ &= \mathbf{P}_{\sigma \sim \mu_h} (\sigma(r) = 0) [\mathbf{P}_{\sigma \sim \mu_h} (F_{h,k}(\sigma) = 1 : \sigma(r) = 1) - \mathbf{P}_{\sigma \sim \mu_h} (F_{h,k}(\sigma) = 1 : \sigma(r) = 0)] \\ &= \mathbf{P}_{\sigma \sim \mu_h} (\sigma(r) = 0) [\alpha_r - \beta_r] \\ &\geq 0. \end{aligned} \tag{20}$$

□

Now, we proceed to analyze this scheme in two cases: For the broadcasting measure ν of the independent sets model and for an approximation $\{\mu_h\}_{h \geq 1}$ of ν through discrete boundary conditions.

2.2.2.1 Analysis for the independent sets measure ν

In this section, ν_h is the broadcasting measure for the independent sets model over the tree T_h (as defined in Example 2.1). In the following theorem, we prove a series of estimates for the ‘parsimonious scheme’ defined above. Such estimates will be used in the analysis of the sensitivity of the scheme (such analysis will be carried out in Section 2.3). In particular, we prove the non-triviality of the scheme in the appropriate regime, implying therefore, the non-extremality of the measure ν . This reproves the result originally stated in [23].

In order to state our result, for $\delta > 0$ and $\omega = (1 + \delta) \ln b/b$ (we recall here that $\lambda = \omega(1 + \omega)^b$), we define

$$b_0(\delta) := \min \left\{ b' : \exp \left(\frac{(1.01)(\omega b)^2}{\lambda} \right) \leq 1.01 \text{ for all } b > b' \right\}. \quad (21)$$

Note that, $b_0(\delta)$ is well-defined, since for any fixed δ ,

$$\lim_{b \rightarrow \infty} \exp \left(\frac{(1.01)(\omega b)^2}{\lambda} \right) = 1.$$

Theorem 2.6. *For $\delta > 0$ and all $b > b_0(\delta)$, setting $\omega = (1 + \delta) \ln b/b$, we have that for any sequence $k := k(h) \leq h$, the following holds:*

1. *For all $h \geq 1$,*

$$\mathbf{P}_{\sigma \sim \nu_h} (F_{h,k}(\sigma) = 0) \leq \frac{(1.01)^{1/b}}{1 + \omega}. \quad (22)$$

2. *For all $h \geq 1$,*

$$\mathbf{P}_{\sigma \sim \nu_h} (F_{h,k}(\sigma) = 0) \geq \frac{1}{1 + \omega} \left(1 - \frac{\ln(1.01)}{(1 + \delta) b \ln b} \right)$$

3. *The scheme is nontrivial.*

4. *The effectiveness of the scheme satisfies*

$$\text{eff}_h(F) \geq \frac{\omega}{(1 + \omega)^2} \left(1 - \frac{0.01}{\omega b} - \frac{(1 + \omega) \ln(1.01)}{(\omega b)^2} \right).$$

5. For the infinite measure ν , there is reconstruction.

Proof. As in the proof of Proposition 2.5, for every $v \in T_h$, we define

$$\alpha_v := \mathbf{P}_{\sigma \sim \nu_h} (G_{h,k}(\sigma, v) = 1 : \sigma(v) = 1)$$

and

$$\beta_v := \mathbf{P}_{\sigma \sim \nu_h} (G_{h,k}(\sigma, v) = 1 : \sigma(v) = 0).$$

Also, define

$$f_v := \frac{\omega}{1+\omega} (1 - \alpha_v) + \frac{1}{1+\omega} (1 - \beta_v).$$

Now, recalling that for the measure ν_h , $m^e = \begin{bmatrix} \frac{1}{1+\omega} & \frac{\omega}{1+\omega} \\ 1 & 0 \end{bmatrix}$ for every edge e in T_h , we get from eqs. (18) and (19) the following recursion: For every vertex $v \in T_h$ at distance $\leq k-2$ from the root, if we denote by w_1, \dots, w_b the children of v and, for every $i = 1, \dots, b$, we denote by $w_{i,1}, \dots, w_{i,b}$ the children of w_i , then

$$\alpha_v = \prod_{i=1}^b \left(1 - \prod_{j=1}^b f_{w_{i,j}}\right), \text{ and } \beta_v = \prod_{i=1}^b f_{w_i}. \quad (23)$$

Therefore,

$$\begin{aligned} f_v &= \frac{\omega}{1+\omega} (1 - \alpha_v) + \frac{1}{1+\omega} (1 - \beta_v) \\ &= 1 - \frac{\omega}{1+\omega} \prod_{i=1}^b \left(1 - \prod_{j=1}^b f_{w_{i,j}}\right) - \frac{1}{1+\omega} \prod_{i=1}^b f_{w_i} \end{aligned} \quad (24)$$

In particular, we have for $v \in L_k$,

$$f_v = \frac{1}{1+\omega} \leq \frac{(1.01)^{1/b}}{1+\omega},$$

and for $v \in L_{k-1}$,

$$f_v = \frac{1}{1+\omega} \left(1 - \left(\frac{1}{1+\omega}\right)^b\right) \leq \frac{(1.01)^{1/b}}{1+\omega}.$$

Now, if we assume that $f_{w_{i,j}} \leq \frac{(1.01)^{1/b}}{1+\omega}$ for every $i, j = 1, \dots, b$, we deduce from (24) the

following:

$$\begin{aligned}
f_v &= 1 - \frac{\omega}{1+\omega} \prod_{i=1}^b \left(1 - \prod_{j=1}^b f_{w_{i,j}}\right) - \frac{1}{1+\omega} \prod_{i=1}^b f_{w_i} \\
&\leq 1 - \frac{\omega}{1+\omega} \left(1 - \frac{1.01\omega}{\lambda}\right)^b \\
&\leq \frac{1 + \frac{1.01\omega^2 b}{\lambda}}{1+\omega}, \quad \text{since } (1-t)^b \geq 1-tb \text{ for } t \leq 1 \\
&\leq \frac{(\exp(1.01(\omega b)^2/\lambda))^{1/b}}{(1+\omega)}, \quad \text{since } (1+t) \leq e^t \\
&\leq \frac{(1.01)^{1/b}}{(1+\omega)}, \quad \text{from the definition of } b_0(\delta).
\end{aligned}$$

Therefore, by induction, we obtain that $f_r = \mathbf{P}_{\sigma \sim \nu_h}(F_{h,k}(\sigma) = 0) \leq \frac{(1.01)^{1/b}}{1+\omega}$. This proves Item **1** of the Theorem.

To prove Item **2** notice that, in view of Eq. (24), we have that

$$\mathbf{P}_{\sigma \sim \nu_h}(F_{h,k}(\sigma) = 0) = f_r \geq \frac{1}{1+\omega} \left(1 - \frac{1.01}{(1+\omega)^b}\right),$$

and then **2** follows by noticing that, for $b \geq b_0(\delta)$,

$$\frac{1.01}{(1+\omega)^b} \leq \frac{\ln(1.01)}{(1+\delta)b \ln b}. \quad (25)$$

Item **3** from the Theorem follows directly from Items **1** and **2**. Now, to prove Item **4**, we first deduce that

$$\text{eff}_h(F) = \mathbf{P}_{\sigma \sim \nu_h}(\sigma(r) = 0) (\mathbf{P}_{\sigma \sim \nu_h}(F_{h,k}(\sigma) = 0 : \sigma(r) = 0) - \mathbf{P}_{\sigma \sim \nu_h}(F_{h,k}(\sigma) = 0))$$

Now, if w_1, \dots, w_b are the children of the root, we have that

$$\mathbf{P}_{\sigma \sim \nu_h}(\sigma(r) = 0 : \sigma(r) = 0) = 1 - \prod_{i=1}^b f_{w_i} \geq 1 - \frac{1.01}{(1+\omega)^b}.$$

Therefore,

$$\text{eff}_h(F) \geq \frac{1}{1+\omega} \left(1 - \frac{1.01}{(1+\omega)^b} - \frac{1.01^{1/b}}{1+\omega}\right). \quad (26)$$

Now, notice that

$$1 - \frac{(1.01)^{1/b}}{1+\omega} \geq \frac{1}{b} \left[\frac{(1+\delta) \ln b - 0.01}{(1+\omega)} \right]. \quad (27)$$

Thus, combining (25) and (27) into (26) we get the result.

Finally, to prove Item **5**, notice that, defining the reconstruction scheme F_h on the infinite tree by using the same recursion, then

$$\mathbf{P}_{\sigma \sim \nu}(F_h(\sigma) = 1) = \mathbf{P}_{\sigma \sim \nu_h}(F_{h,h}(\sigma) = 1),$$

and similarly,

$$\mathbf{P}_{\sigma \sim \nu}(F_h(\sigma) = 1 \text{ and } \sigma_r = 1) = \mathbf{P}_{\sigma \sim \nu_h}(F_{h,h}(\sigma) = 1 \text{ and } \sigma(r) = 1).$$

Therefore, the effectiveness of the scheme satisfies the bound stated in Item **4**. Thus, in view of Proposition 2.2, there is no decay of correlation between $\sigma(r)$ and σ_{L_h} as $h \rightarrow \infty$, that is, reconstruction holds. \square

2.2.2.2 Analysis for a discrete approximation of ν

Now, we proceed to replicate the results from the previous section in the case of an appropriate discrete approximation to the measure ν . Such analysis will be used in Section 2.3.3 to prove that a specific sequence of discrete boundary conditions slows down the dynamics. In this section, for any ω and b , we will assume that there exists a sequence of **discrete** boundary conditions $\{\Gamma_h\}_{h \geq 1}$ that satisfies the following:

Condition 2.7. *Let μ_h be the Gibbs' measure for the independent sets model in T_h , with boundary condition Γ_h . For every $\epsilon > 0$, there exists $l(\epsilon)$ (not depending on h , but maybe on b and ω), such that for every vertex v with $d(v, r) \leq h - l(\epsilon)$,*

$$\left| \frac{\text{Prob}_{\sigma \sim \mu_{h,v}}(\sigma_v = 0)}{1/(1+\omega)} - 1 \right| \leq \epsilon, \quad (28)$$

(recall that $\mu_{h,v}$ is a short for $\mu_{T_h^{(v)}, \Gamma_h^{(v)}}$).

The existence of a sequence of discrete boundary conditions that satisfies the previous condition is not an easy fact. In Section 5 of our research article [97], for every $\omega < 1$ and $b \geq 1$, a sequence of boundary conditions with the above property is constructed.

Now, in order to state our result, for $\delta > 0$ and $\omega = (1 + \delta) \ln b/b$, we define

$$b_0(\delta) := \min\{b' : \exp\left(\frac{2(1.01)(\omega b')^2}{\lambda}\right) \leq 1.01 \text{ for all } b > b'\}. \quad (29)$$

This function is well defined just as the one in Eq. (21) (notice, however, the slight difference between both definitions).

The following result is analogous to Theorem 2.6.

Theorem 2.8. *For $\delta > 0$ and $b > b_0(\delta)$, let us set $\omega = (1 + \delta) \ln b / b$. If the sequence of measures $\{\mu_h\}_{h \geq 1}$ satisfies Condition 2.7, then there is some l (not depending on h) such that for any sequence $k := k(h) \leq h - l$, the following holds:*

1. For all $h \geq 1$,

$$\mathbf{P}_{\sigma \sim \mu_h}(F_{h,k}(\sigma) = 0) \leq \frac{(1.01)^{1/b}}{1 + \omega}.$$

2. For all $h \geq 1$,

$$\mathbf{P}_{\sigma \sim \mu_h}(F_{h,k}(\sigma) = 0) \geq \frac{1}{1 + \omega} \left(1 - \frac{\ln(1.01)}{2(1 + \delta)b \ln b} \right)$$

3. The reconstruction scheme is nontrivial.

4. The effectiveness of the scheme is

$$\text{eff}_h(F) \geq \frac{\omega}{(1 + \omega)^2} \left(1 - \frac{0.01}{\omega b} - \frac{(1 + \omega) \ln(1.01)}{2(\omega b)^2} \right).$$

Proof. For $\epsilon = \left\lceil \exp\left(\frac{1.01\omega^2 b}{\lambda}\right) - 1 \right\rceil$, take $l = l(\epsilon)$ such that (28) holds. Let us recall the following definitions from the proof of Proposition 2.5. For every $v \in T_h$, let

$$\alpha_v := \mathbf{P}_{\sigma \sim \mu_h}(G_{h,k}(\sigma, v) = 1 : \sigma(v) = 1)$$

and

$$\beta_v := \mathbf{P}_{\sigma \sim \mu_h}(G_{h,k}(\sigma, v) = 1 : \sigma(v) = 0).$$

Also, for $v \in T_h$, define

$$f_v = \text{Prob}_{\sigma \sim \mu_{h,v}}(\sigma_v = 1)(1 - \alpha_v) + \text{Prob}_{\sigma \sim \mu_{h,v}}(\sigma(v) = 0)(1 - \beta_v).$$

Now, the proof proceeds in a way entirely analogous to the proof of Theorem 2.6. From Eqs. (18) and (19) we get that, for every vertex $v \in T_h$ at distance $\leq k - 2$ from the

root, if we denote by w_1, \dots, w_b the children of v and, for every $i = 1, \dots, b$, we denote by $w_{i,1}, \dots, w_{i,b}$ the children of w_i , then

$$\alpha_v = \prod_{i=1}^b \left(1 - \prod_{j=1}^b f_{w_{i,j}}\right), \text{ and } \beta_v = \prod_{i=1}^b f_{w_i}.$$

Therefore,

$$\begin{aligned} f_v &= \text{Prob}_{\sigma \sim \mu_{h,v}} (\sigma(v) = 1) (1 - \alpha_v) + \text{Prob}_{\sigma \sim \mu_{h,v}} (\sigma(v) = 0) (1 - \beta_v) \\ &= 1 - \text{Prob}_{\sigma \sim \mu_{h,v}} (\sigma_v = 1) \prod_{i=1}^b \left(1 - \prod_{j=1}^b f_{w_{i,j}}\right) - \text{Prob}_{\sigma \sim \mu_{h,v}} (\sigma(v) = 0) \prod_{i=1}^b f_{w_i} \end{aligned}$$

Now, if v is such that $d(v, r) \leq k(h) - 2 \leq h - l$, we have that

$$f_v \leq \left(1 - \frac{\omega}{1 + \omega} \prod_{i=1}^b \left(1 - \prod_{j=1}^b f_{w_{i,j}}\right) - \frac{1}{1 + \omega} \prod_{i=1}^b f_{w_i}\right) \exp\left(\frac{1.01\omega^2 b}{\lambda}\right).$$

In particular, notice that, for $v \in L_{k(h)}$, we have that

$$f_v = \text{Prob}_{\sigma \sim \mu_{h,v}} (\sigma_v = 0) \leq \frac{1}{1 + \omega} \exp\left(\frac{1.01\omega^2 b}{\lambda}\right) \leq \frac{(1.01)^{1/b}}{1 + \omega} \quad (\text{for } b \geq b_0(\delta)),$$

and, for $v \in L_{k(h)-1}$,

$$f_v \leq \text{Prob}_{\sigma \sim \mu_{h,v}} (\sigma_v = 0) \leq \frac{(1.01)^{1/b}}{1 + \omega} \quad (\text{by the same argument}).$$

Now (with the above notation), if we assume that $f_{w_{i,j}} \leq \frac{(1.01)^{1/b}}{1 + \omega}$ for $i, j = 1, \dots, b$, then,

$$\begin{aligned} f_v &\leq \left(1 - \frac{\omega}{1 + \omega} \left(1 - \frac{1.01}{(1 + \omega)^b}\right)^b\right) \exp\left(\frac{1.01\omega^2 b}{\lambda}\right) \\ &\leq \frac{(\exp(1.01(\omega b)^2/\lambda))^{1/b}}{(1 + \omega)} \exp\left(\frac{1.01\omega^2 b}{\lambda}\right), \quad \text{as in the proof of Theorem 2.6} \\ &= \frac{(\exp(2(1.01)(\omega b)^2/\lambda))^{1/b}}{(1 + \omega)} \\ &\leq \frac{(1.01)^{1/b}}{(1 + \omega)}, \quad \text{from the definition of } b_0(\delta). \end{aligned}$$

Therefore, by induction, we get that $f_v \leq \frac{(1.01)^{1/b}}{1 + \omega}$ for every v at distance $\leq k(h) \leq h - l$ from the root, and in particular, $f_r = \text{Prob}_{\sigma \sim \mu_h} (F_h(\sigma) = 0) \leq \frac{(1.01)^{1/b}}{1 + \omega}$. This proves Item **1**.

Once Item **1** is established, Items **2**, **3** and **4**, follow exactly in the same way as in Theorem 2.6. \square

2.3 *Independent sets model on regular trees: A dynamical phase transition*

In view of the various results that connect the notion of extremality with the mixing time of the Glauber dynamics on trees and, in particular, the tight phase transition exhibited in [108] at the extremality transition for colorings, we expect that such a relation is still present in the independent sets model on trees. In fact, several steps have been taken, for both, general graphs and trees, to investigate such a connection. In [34, 68, 112], the authors proved that in graphs with maximum degree $b+1$, for $\lambda \leq \frac{2}{b-1}$, the Glauber dynamics mixes in $O(n \log n)$ steps. This is a result that cannot be improved too much for general graphs: It is the case that, for $\lambda > \lambda_{\text{uniq}} = \frac{b^b}{(b-1)^{b+1}} \approx \frac{e}{b}$, approximate sampling of instances of the independent sets model in polynomial time is not possible unless $\text{NP}=\text{RP}$ (see [38, 82, 102]). Nevertheless, in the case of trees, the situation is a bit more different, in particular, the recursion in Eq. (14) permits the sampling of instances in polynomial time. Moreover, it is known that the Glauber dynamics mixes in polynomial time for regular boundary conditions [13, 75]. Therefore, in the case of trees, the phase transition we are looking, instead, is a discontinuity in the polynomial rate of the relaxation time. More exactly, a phase transition for the mixing time of the Glauber dynamics occurs, if the quantity $\frac{\log(\tau_{\text{rel}})}{\log n}$ exhibits a discontinuity at a certain value of the parameter λ , when λ grows over a given scale.

Notice, in particular, that the result for $\lambda \leq \frac{2}{b-1}$ mentioned above, does not cover the whole uniqueness region in the case of trees. On the other hand, Martinelli, Sinclair and Weitz [75] proved rapid mixing (that is, mixing in $O(n \log n)$ steps) for arbitrary boundary conditions in a regime that goes half way into the extremality region. This again suggests that the appropriate phase transition for the mixing time must be located at the point where a phase transition for extremality occurs. Their precise result is the following (Theorem 1.2 from [75]).

Theorem 2.9 ([75]). *For the independent sets model on the n -vertex, b -ary tree T_h with boundary condition Γ_h , the mixing time of the Glauber dynamics is $O(n \log n)$ in both of the following situations:*

1. The boundary condition Γ_h is arbitrary, and $\lambda < \max \left\{ \left(\frac{b+1}{b-1} \right)^2 - 1, \frac{1}{\sqrt{b-1}} \right\}$.
2. The boundary condition Γ_h is even (or odd), and λ is arbitrary.

Moreover, in both of the above cases the limiting Gibbs' measure is extremal.

In [75], it remained an intriguing open problem to establish a sharp threshold for the mixing time in the independent sets model. In particular, the authors asked if there is a sequence of (discrete) boundary conditions for which the single-site Glauber dynamics for the independent sets model slows down. The following theorem answers these questions.

Theorem 2.10. *The single site Glauber dynamics for the independent sets model on the tree T_h and boundary condition Γ_h , satisfies the following:*

1. For $\omega \leq \ln b/b$ and arbitrary Γ_h ⁷

$$\tau_{rel} \leq C_{\delta,b} n^{1+\epsilon(b)},$$

where $\epsilon(b) \leq c \ln \ln b / \ln b$.

2. For all $\delta > 0$, $\omega = (1 + \delta) \ln b/b$ and arbitrary Γ_h ,

$$\tau_{rel} \leq C_{\delta,b} n^{1+\delta+\epsilon(b)}.$$

where $\epsilon(b) \leq c \ln \ln b / \ln b$.

In the following, I is a fixed interval $I = (0, R)$, where R is a fixed arbitrary constant

3. For all $\delta \in I$, if $\omega = (1 + \delta) \ln b/b$ and Γ_h is the boundary such that $\psi_v^{(\Gamma_h)}(0) = \frac{1}{1+\omega}$ and $\psi_v^{(\Gamma_h)}(1) = \frac{\omega}{1+\omega}$,

$$\tau_{rel} \geq C_{\delta,b} n^{1+\delta/2+\epsilon(b)},$$

where $\epsilon(b) \leq c \ln \ln b / \ln b$.

⁷Recall that C , C' , c , c' , etc. are numerical constants. We explicitly state the parameters they depend on. For instance $C_{\delta,b}$ is a numerical constant that depends only on δ and b , and c is a fixed real number.

4. For all $\delta \in I$, $\omega = (1 + \delta) \ln b/b$, there exists a sequence of **discrete** boundary conditions $\{\Gamma_h\}_{h \geq 1}$, such that

$$\tau_{\text{rel}} \geq C_{\delta,b} n^{1+\delta/2+\epsilon(b)},$$

where $\epsilon(b) \leq c \ln \ln b / \ln b$.

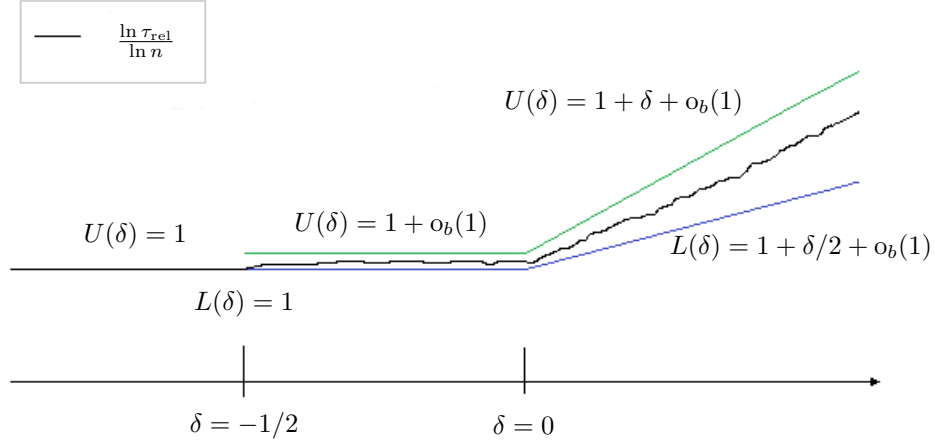


Figure 2: Mixing time of the Glauber dynamics for the independent sets model with $\omega = (1 + \delta) \ln b/b$ on the b -regular tree. Above, it is the case that $L(\delta) \leq \frac{\ln \tau_{\text{rel}}}{\ln n} \leq U(\delta)$

Remark 2.11. A tool that permits to obtain a tighter bound for mixing time than the one obtained by a raw comparison with the spectral gap, as in Eq. (7), is the so-called log-sobolev constant of the Markov process. Given a nonnegative function $f : \Omega \rightarrow \mathbb{R}^+$ and a measure μ over Ω , we define the entropy of f respect to μ , as

$$\text{Ent}_\mu(f) = \mu(f \log f) - \mu(f) \log \mu(f).$$

Now, the log-sobolev constant c_{sob} of a Markov process is defined as

$$c_{\text{sob}} = \inf_{f \geq 0} \frac{\mathcal{D}(\sqrt{f})}{\text{Ent}_\mu(f)},$$

where μ is its stationary measure, and \mathcal{D} is the Dirichlet form as defined in Eq. (8). Now, with the use of the log-sobolev constant, we obtain the following tighter estimate for the mixing time (see [55, 79]) (compare with Eq. (7))

$$\tau_{\text{mix}}(x, \epsilon) = O\left(\frac{1}{c_{\text{sob}}} \left(\ln \ln \frac{1}{\pi(x)}\right)\right). \quad (30)$$

For 2-spin systems on trees, the authors in [74] (see also [108]), established a close relation between the spectral gap and the log-sobolev constant of the single-site Glauber dynamics in trees. In particular, they proved that

$$c_{sob}^{-1} \leq c_{gap}^{-1} \times C \log n. \quad (31)$$

Combining this relation with the bound in Eq. (30) we can translate the results of Theorem 2.10 in terms of mixing time instead of relaxation time. The result is identical, just replacing τ_{rel} by τ_{mix} (we can hide the $(\log n)^2$ loss from the comparison in Eqs. (30) and (31), inside the small order term of the exponent).

Remark 2.12. We should also point out that an analysis of the Dirichlet form of the chain, following [13], leads to the following lower bound for the relaxation time of the Glauber dynamics for the simple invariant version ν_h ,

$$\tau_{rel} \geq C_{\omega,b} n^{1+\log_b(b\theta^2)}, \quad (32)$$

where $\theta = \frac{\omega}{1+\omega}$ is the magnitude of the second eigenvalue of the broadcasting matrix m . Such a bound is not trivial only for $b\theta^2 > 1$, exhibiting also a phase transition at the so-called census reconstruction threshold [81]. Notice, however, that the result from Theorem 2.10 is clearly stronger.

2.3.1 Upper bound for arbitrary boundary conditions: Proof of part 1 and 2.

To establish this bound, we will use the *block dynamics* approach of Martinelli [74]. Through this approach, the question of establishing an upper bound for the mixing time of the Glauber dynamics on the b -ary tree T_h with arbitrary boundary condition, is reduced to establishing a bound for the mixing time of the Glauber dynamics on the tree of height 1, T_1 , (that is, the star graph). More exactly, let $\tau_{rel}^* := \max_{\Gamma} \left\{ \tau_{rel}^{(\Gamma)} \right\}$, where $\tau_{rel}^{(\Gamma)}$ is the relaxation time of the Glauber dynamics of the independent sets model in T_1 , with boundary condition Γ (not necessarily discrete). Then, the relaxation time for the Glauber dynamics of the independent sets model in T_h with (arbitrary) boundary condition Γ_h , satisfies

$$\tau_{rel} \leq (\tau_{rel}^*)^h. \quad (33)$$

This approach was used, in the context of trees, in [13, 69, 108].

Therefore, in view of Eq. (33), the following lemma will be enough to deduce the result.

Lemma 2.13. *For any boundary condition Γ , the mixing time of the Glauber dynamics for the independent sets measure on the tree T_1 , with boundary condition Γ , satisfies*

$$\tau_{\text{mix}} \leq \lceil \lambda + 1 \rceil (b + 1) \ln(4e(b + 1)) \ln(64e^2 L^2 (b + 1)^3) \quad (34)$$

We will prove this lemma at the end of the section.

Proof of the Upper bound. In view of Eq. (34) we have, in particular in page 8), that

$$\tau_{\text{rel}}^* \leq \lceil \lambda + 1 \rceil (b + 1) \ln(4e(b + 1)) \ln(64e^2 L^2 (b + 1)^3) + 1.$$

Therefore, the Glauber dynamics of the independent sets model in T_h with (arbitrary) boundary condition Γ_h , satisfies

$$\tau_{\text{rel}} \leq \left(\lceil \lambda + 1 \rceil (b + 1) \ln(4e(b + 1)) \ln(64e^2 L^2 (b + 1)^3) + 1 \right)^h \leq n^d,$$

where

$$d = 1 + \frac{\ln \left(\lceil \lambda + 1 \rceil (1 + 1/b) \ln(4e(b + 1)) \ln(64e^2 L^2 (b + 1)^3) + 1/b \right)}{\ln(b)}.$$

Now, if $\omega \leq \frac{\ln b}{b}$, we have that, for some constant c ,

$$d \leq 1 + c \ln \ln b / \ln b.$$

On the other hand, for $\delta > 0$ and $\omega = (1 + \delta) \ln b / b$, we instead get, that for some constant c ,

$$d \leq 1 + \delta + \frac{c \ln \ln b}{\ln b}.$$

□

Now, we will proceed to prove Lemma 2.13.

Let r be the root of the star graph T_1 and let w_1, \dots, w_b be the children of r . Given a pair of configurations $\eta, \eta' : T_1 \rightarrow \{0, 1\}$, we say that $\eta \leq \eta'$ if $\eta(r) \leq \eta'(r)$ and, for

every $i = 1, \dots, b$, $\eta(w_i) \geq \eta'(w_i)$ ⁸. Let η_{\max} and η_{\min} be the unique maximal and minimal elements in this order, respectively.

Let $X = \{X_t\}_{t \geq 0}$, with $X_0 = \eta_{\max}$ be the Glauber dynamics for the independent sets model on T_1 . Given a sequence $u = (u_1, u_2, \dots)$ of vertices of T_1 , let $X^{(u)} = \{X_t^{(u)}\}_{t \geq 0}$ be the Glauber dynamics such that $X_0 = \eta_{\max}$ and, for every $t \geq 1$, the chain is restricted to effectuate the Glauber update at the vertex u_t at time t .

In the following, $\mathbf{U} = (\mathbf{u}_1, \mathbf{u}_2, \dots)$ is a sequence of i.i.d uniform random vertices of T_1 . Notice that the (original) Glauber dynamics X satisfies

$$X \stackrel{d}{=} X^{(\mathbf{U})}. \quad (35)$$

Given a 0/1 sequence $\gamma = (\gamma_t)_{t \geq 1}$, and a sequence of vertices $u = (u_1, u_2, \dots)$, we define $X^{(u, \gamma)} = \{X_t^{(u, \gamma)}\}_{t \geq 0}$ to be the *censored* version of $X^{(u)}$, which is restricted, in addition, to make Glauber updates only at times t such that $\gamma_t = 1$.

Analogously, we define the processes Y , $Y^{(u)}$ and $Y^{(u, \gamma)}$ starting, instead, at η_{\min} . The ‘censoring lemma’ of Peres and Winkler (the result remains unpublished but its proof can be found in [86]), states that⁹

$$Y_t^{(u, \gamma)} \preceq Y_t^{(u)} \preceq X_t^{(u)} \preceq X_t^{(u, \gamma)}. \quad (36)$$

Now, let $\mu_{X_t^{(u)}}$, $\mu_{X_t^{(u, \gamma)}}$, $\mu_{Y_t^{(u)}}$ and $\mu_{Y_t^{(u, \gamma)}}$ be the distributions of $X_t^{(u)}$, $X_t^{(u, \gamma)}$, $Y_t^{(u)}$ and $Y_t^{(u, \gamma)}$ respectively. Due to the fact that $\mu_{X_t^{(u)}}/\mu_{Y_t^{(u)}}$ is increasing (also established in [86]), from Eq. (36) we have that

$$\text{TV} \left(X_t^{(u)}, Y_t^{(u)} \right) \leq \text{TV} \left(X_t^{(u, \gamma)}, Y_t^{(u, \gamma)} \right). \quad (37)$$

Moreover, Eqs. (36) and (37) still hold true if the initial condition for $X_t^{(u)}$ is replaced by a random initial condition X_0 such that its distribution, μ_{X_0} , is such that μ_{X_0}/μ is increasing; and, the initial condition for $Y_t^{(u)}$ is replaced by a random initial condition Y_0 such that its distribution, μ_{Y_0} , is such that μ_{Y_0}/μ is increasing.

⁸Such order makes the independent sets model over T_1 (or more generally, over bipartite graphs), a *monotone* system (see [65, Chapter 2], for example).

⁹Recall the definition of stochastic dominance from page 1.

Proof of Lemma 2.13. Let $L := \lceil \lambda + 1 \rceil \ln(4e(b+1))$ and $T := (t' + t'')L$, where

$$t' := (b+1) \ln(8eL(b+1))$$

and

$$t'' := (b+1) \ln(8eL(b+1)^2)$$

(notice that T is the aimed bound on the mixing time). Given a sequence of vertices $u = \{u_t\}_{t \geq 1}$, we say that the sequence is ‘good’, if there exist $0 < t_1 < t_2 < \dots < t_L < t_{L+1} = T$ such that:

1. $u_{t_i} = r$ for every $i = 1, \dots, L$.
2. For every $i = 1, \dots, L$ and every $j = 1, \dots, b$, there exists $t \in \{t_i + 1, \dots, t_{i+1}\}$ such that $u_t = w_i$. In such a case, we define,

$$t_{i,j} := \min \{t \in \{t_i + 1, \dots, t_{i+1}\} : u_t = w_i\}.$$

Now, given a ‘good’ sequence of vertices $u = (u_1, u_2, \dots)$, let γ be the censoring such that $\gamma_t = 1$ iff $t = t_i$ for some $i = 1, \dots, L$ or $t = t_{i,j}$ for some $i = 1, \dots, L$ and $j = 1, \dots, b$. For this specific censoring (defined only if the sequence is ‘good’), let us $(\bar{X}_t^{(u,\gamma)}, \bar{Y}_t^{(u,\gamma)})$ be the monotone coupling of the processes $X_t^{(u,\gamma)}$ and $Y_t^{(u,\gamma)}$. A straightforward observation is that $\bar{X}_T^{(u,\gamma)} = \bar{Y}_T^{(u,\gamma)}$ unless, for every $i = 1, \dots, L$, the root is assigned to value 1 by the dynamics. Therefore, in view of Eq. (37),

$$\begin{aligned} \text{TV} \left(X_T^{(u)}, Y_T^{(u)} \right) &\leq \text{TV} \left(X_T^{(u,\gamma)}, Y_T^{(u,\gamma)} \right) \\ &\leq \mathbf{P} \left(\bar{X}_T^{(u,\gamma)} \neq \bar{Y}_T^{(u,\gamma)} \right) \\ &\leq \left(1 - \frac{1}{1+\lambda} \right)^L \\ &\leq \frac{1}{4e(b+1)} \end{aligned}$$

Now, the following holds for the random sequence $\mathbf{U} = (\mathbf{u}_1, \mathbf{u}_2, \dots)$. Let G is a geometric random variable with mean $b+1$, and C is the time to collect $b+1$ coupons (in the language

of the coupon collector problem).

$$\begin{aligned}
\mathbf{P}(\mathbf{U} \text{ is good}) &\geq (\mathbf{P}(G \leq t') \mathbf{P}(C \leq t''))^L \\
&\geq \left(1 - \left(1 - \frac{1}{b+1}\right)^{t'}\right)^L \left(1 - (b+1)^{-\frac{t''}{(b+1)\ln(b+1)}+1}\right)^L \\
&\geq 1 - \frac{1}{4e(b+1)}.
\end{aligned}$$

Therefore, for the processes X and Y we get, from (35), by conditioning on \mathbf{U} , that

$$\text{TV}(\mu_{X_T}, \mu_{Y_T}) \leq \mathbf{P}(\mathbf{U} \text{ is not good}) + \frac{1}{4e(b+1)} \leq \frac{1}{2e(b+1)}. \quad (38)$$

Now, let us consider the monotone coupling $\{(\bar{X}_t, \bar{Y}_t)\}_{t \geq 0}$ of the processes X_t and Y_t . For every $v \in T_1$, $\bar{X}_t(v) \geq \bar{Y}_t(v)$, and therefore.

$$\mathbf{P}(\bar{X}_t(v) > \bar{Y}_t(v)) = \mathbf{P}(\bar{X}_t(v) = 1) - \mathbf{P}(\bar{Y}_t(v) = 1) \leq \text{TV}(\mu_{X_t}, \mu_{Y_t}).$$

Now,

$$\mathbf{P}(\bar{X}_T > \bar{Y}_T) \leq \sum_{v \in T_1} \mathbf{P}(\bar{X}_T(v) > \bar{Y}_T(v)) \leq (b+1) \text{TV}(\mu_{X_{t^*}}, \mu_{Y_{t^*}}) \leq \frac{1}{2e}.$$

From where the result follows in view of Eq. (6) in page (8) □

2.3.2 Slow down in the Glauber dynamics for the measure ν : Proof of part 3 of Theorem 2.10

In order to prove part **3**, we will analyze the ‘sensitivity’ of the parsimonious reconstruction scheme described in Section 2.2.2 (whose analysis for the measure ν was included in Section 2.2.2.1). Now, for $v \in T_h$ and $\eta \in \Omega_{T_h}$, define $\eta^{(v)}$ to be the configuration that differs from η exactly at the vertex v . We have that,

$$\begin{aligned}
\mathbf{E}_{\sigma \sim \nu_h} [\nabla F_{h,k}(\sigma)] &= \frac{1}{|L_k|} \mathbf{E}_{\sigma \sim \nu_h} \left[\sum_{u \in L_k} \mathbf{I}(F_{h,k}(\sigma) = 1 \text{ and } F_{h,k}(\sigma^{(u)}) = 0) \right] \\
&= \frac{1}{|L_k|} \sum_{u \in L_k} \mathbf{P}_{\sigma \sim \nu_h} (F_{h,k}(\sigma) = 1 \text{ and } F_{h,k}(\sigma^{(u)}) = 0),
\end{aligned}$$

and due to the invariance of the measure ν_h ,

$$\mathbf{E}_{\sigma \sim \nu_h} [\nabla F_{h,k}(\sigma)] = \mathbf{P}_{\sigma \sim \nu_h} (F_{h,k}(\sigma) = 1 \text{ and } F_{h,k}(\sigma^{(u^*)}) = 0), \quad (39)$$

where $u^* \in L_k$ is a fixed vertex.

We will now look at bounding $\mathbf{P}_{\sigma \sim \nu_h} (F_{h,k}(\sigma) = 1 \text{ and } F_{h,k}(\sigma^{(u^*)}) = 0)$. For this, let $\mathbf{p} = u_0, u_1, \dots, u_k$, with $u_0 = r$ and $u_k = u^*$, be the path joining the root with the vertex u^* . Now, notice the following:

Observation 1. *Given $\eta \in \Omega_{T_h}$, the following two conditions are necessary and sufficient in order for η to satisfy $F_{h,k}(\eta) = 1$ and $F_{h,k}(\eta^{(u^*)}) = 0$:*

1. $G_{h,k}(\eta, w) = 0$ for every $w \notin \mathbf{p}$ such that $w \sim u_i$ for some $i = 0, \dots, k-1$.¹⁰
2. $\eta(u^*) = 1$ if h is even, $\eta(u^*) = 0$ if h is odd.

This fact allows us to prove the following lemma.

Lemma 2.14. *For every i , let $f_i := \mathbf{P}_{\sigma \sim \nu_{h-i}}(F_{h-i,k-i}(\sigma) = 0)$. Then,*

$$\mathbf{P}_{\sigma \sim \nu_h} (F_{h,k}(\sigma) = 1 \text{ and } F_{h,k}(\sigma^{(u^*)}) = 0) \leq \mathbf{E}_{\sigma \sim \nu_h} \left[\prod_{i=1}^k (f_{i+1}^{b-1})^{1-\sigma(u_i)} \right]$$

where $\mathbf{p} = u_0, u_1, \dots, u_k$, with $u_0 = r$ and $u_k = u^*$, is the path joining the root and u^* .

Proof. For every $i = 1, \dots, k$, let $w_{i,1}, \dots, w_{i,b-1}$ be the children of the vertex u_i that are not in the path \mathbf{p} . Define also, for $v \in T_h$ such $d(v, r) = i$, $f_v := \mathbf{P}_{\sigma \sim \nu_{h,v}}(F_{h-i,k-i}(\sigma) = 0)$ (recall that $\nu_{h,v}$ is a short for $\nu_{T_h^{(v)}, \Gamma_h^{(v)}}$). Notice that, due to the invariance of ν_h , it is the case that $f_v = f_i$ for every $v \in L_i$. Now, using the equivalent definition of the event $\{\eta : F_{h,k}(\eta) = 1 \text{ and } F_{h,k}(\eta^{(u^*)}) = 0\}$ indicated previously, we have that

$$\begin{aligned} & \mathbf{P}_{\sigma \sim \nu_h} (F_{h,k}(\sigma) = 1 \text{ and } F_{h,k}(\sigma^{(u^*)}) = 0) \\ & \leq \mathbf{P}_{\sigma \sim \nu_h} (G_{h,k}(\sigma, w_{i,j}) = 0 \text{ for } i \in 0, \dots, k-1, j = 1, \dots, b-1). \end{aligned}$$

Now, to calculate the above probability, we expose the configurations along the path \mathbf{p} . For this purpose, given a configuration $\zeta : \{0, \dots, k\} \rightarrow \{0, 1\}$, let $\alpha_\zeta := \mathbf{P}_{\sigma \sim \nu_h}(\sigma(u_i) = \zeta_i \text{ for } i = 0, \dots, k)$.

¹⁰ Otherwise, it would be the case that $G_h(\sigma, u_i) = G_h(\sigma^{(u^*)}, u_i)$, and therefore $F_h(\sigma) = F_h(\sigma^{(u^*)})$, which is a contradiction.

$i = 0, \dots, k$). Now,

$$\begin{aligned}
& \mathbf{P}_{\sigma \sim \nu_h} (F_{h,k}(\sigma) = 1 \text{ and } F_{h,k}(\sigma^{(u^*)}) = 0) \\
& \leq \sum_{\zeta} \alpha_{\zeta} \prod_{i=0}^{k-1} \prod_{j=1}^{b-1} \mathbf{P}_{\sigma \sim \nu_h} (G_{h,k}(\sigma, w_{i,j}) = 0 : \sigma(u_i) = \zeta_i) \\
& \leq \sum_{\zeta} \alpha_{\zeta} \prod_{\substack{i=0, \dots, k-1 \\ \zeta_i=0}} \prod_{j=1}^{b-1} \mathbf{P}_{\sigma \sim \nu_h} (G_{h,k}(\sigma, w_{i,j}) = 0 : \sigma(u_i) = 0) \\
& = \sum_{\zeta} \alpha_{\zeta} \prod_{\substack{i=0, \dots, h-1 \\ \zeta_i=0}} \prod_{j=1}^{b-1} \mathbf{P}_{\sigma \sim \nu_{h,w_{i,j}}} (F_{h-i-1,k-i-1}(\sigma) = 0) \\
& = \sum_{\zeta} \alpha_{\zeta} \prod_{\substack{i=0, \dots, k-1 \\ \zeta_i=0}} (f_{i+1})^{b-1} \\
& \leq \mathbf{E}_{\sigma \sim \nu_h} \left[\prod_{i=0}^{k-1} \left[(f_{i+1})^{b-1} \right]^{1-\sigma(u_i)} \right].
\end{aligned}$$

□

Now, using the bound on f_i from Theorem 2.6 (Eq. (22)), we are able to combine the above discussion together to prove part 3 of Theorem 2.10:

Proof of part 3 of Theorem 2.10. We can obtain an upper bound on the average sensitivity of the parsimonious scheme $F_{h,k}$ in the following way:

$$\begin{aligned}
\mathbf{E}_{\sigma \sim \nu_h} [\nabla F_{h,k}(\sigma)] &= \mathbf{P}_{\sigma \sim \nu_h} (F_{h,k}(\sigma) = 1 \text{ and } F_{h,k}(\sigma^{(u^*)}) = 0) \quad \text{by Eq. (39)} \\
&\leq \mathbf{E}_{\sigma \sim \nu_h} \left[\prod_{i=1}^k \left((f_{i+1})^{b-1} \right)^{1-\sigma(u_i)} \right] \quad \text{by Lemma 2.14} \\
&\leq \mathbf{E}_{\sigma \sim \nu_h} \left[\left(\frac{(1.01)^{1/b}}{1+\omega} \right)^{(b-1)\#\{i \leq k-1 : \sigma(u_i)=0\}} \right] \quad \text{by Item 1 in Theorem 2.6} \\
&\leq \mathbf{E}_{\sigma \sim \nu_h} \left[\left(\frac{1.01\omega(1+\omega)}{\lambda} \right)^{N_{\mathbf{p}}(\sigma)} \right], \text{ where } N_{\mathbf{p}}(\sigma) = \#\{i \leq k-1 : \sigma(u_i) = 0\}.
\end{aligned}$$

Now, in order to bound this expectation we make use of a delicate analysis of $N_{\mathbf{p}}(\sigma)$ that is stated in the following lemma.

Lemma 2.15. *For all $\delta > 0$ and all $b > \max\{b_0(\delta), 23153\}$ (recall the definition of $b_0(\delta)$)*

from Eq. (21)), setting $\omega = (1 + \delta) \ln b/b$, we have,

$$\mathbf{E}_{\sigma \sim \nu_h} \left[\left(\frac{1.01\omega(1+\omega)}{\lambda} \right)^{N_{\mathbf{p}}(\sigma)} \right] \leq C_{b,\delta} \left(\frac{1.01\omega}{\lambda^{1/2}} \right)^k.$$

Lemma 2.15 is proved at the end of this section. Now, we take $k = k(h) := h$. Then, by the fact that $n = \frac{b^{h+1}-1}{b-1}$, we have, for $\delta > 0$ and $b > \max\{b_0(\delta), 23153\}$, setting $\omega = (1 + \delta) \ln b/b$,

$$\mathbf{E}_{\sigma \sim \nu_h} [\nabla F_{h,k}(\sigma)] = C_{b,\delta} \left(\frac{1.01\omega}{\lambda^{1/2}} \right)^h \leq C'_{b,\delta} \left(n^{-\left[1 + \frac{\ln(\lambda/(1.01\omega b)^2)}{2 \ln b}\right]} \right).$$

Now, from the fact that the scheme $F_{h,k}$ is nontrivial for $\delta > 0$ and $b > b_0(\delta)$, where $\omega = (1 + \delta) \ln b/b$ (from part 3 of Theorem 2.6), Theorem 2.4 applies. Therefore, we get that for $\delta > 0$ and $b \geq \max\{b_0(\delta), 23153\}$, setting $\omega = (1 + \delta) \ln b/b$,

$$\tau_{\text{rel}} \geq C''_{b,\delta} n^d, \quad \text{where } d = \left(1 + \frac{\ln(\lambda/(1.01\omega b)^2)}{2 \ln b} \right).$$

Now, the result is a straightforward corollary by noticing that, for all b and $\delta \in I$, we have that $d \geq 1 + \delta/2 - c \frac{\ln \ln b}{\ln b}$ for some positive constant c . Also, note that, when $b < \max\{b_0(\delta), 23153\}$, we can use the trivial bound $\tau_{\text{rel}} \geq n$, so that taking $\epsilon(b) = c' \frac{\ln \ln b}{\ln b}$ for an appropriate constant c' , the result follows. \square

Remark 2.16. We also could have used the trivial lower bound $N_{\mathbf{p}}(\sigma) \geq h/2$. This leads to the bound $\mathbf{E}_{\sigma \sim \nu_h} [\nabla F_{h,k}(\sigma)] \leq \left(\frac{1.01\omega(1+\omega)}{\lambda} \right)^{h/2}$. Therefore, from Theorem 2.4, we get that $\tau_{\text{rel}} \geq \frac{C_{b,\delta}}{\mathbf{E}_{\sigma \sim \nu_h} [\nabla F_{h,k}(\sigma)]} \approx C_{b,\delta} n^{(1+\delta)/2}$ for the mixing time, which shows a phase transition, although weaker than the one stated in part 3 of Theorem 2.10.

Now, we proceed to prove Lemma 2.15. For its proof, we need the following lemma.

Lemma 2.17. Let ζ_0, ζ_1, \dots be a Markov process with state space $\{0, 1\}$, such that $\zeta_0 = 0$, and with transition rates $p_{0 \rightarrow 0} = p$, $p_{0 \rightarrow 1} = q$, $p_{1 \rightarrow 0} = 1$, $p_{1 \rightarrow 1} = 0$. Let $N_h := \#\{1 \leq i \leq h : \zeta_i = 0\}$. Then, for any $z > 0$:

$$\mathbf{E} [z^{N_h}] \leq C_{p,z} \left(\frac{pz}{2} \left[1 + \sqrt{1 + 4q/(zp^2)} \right] \right)^h,$$

for some constant C_p depending only on p and z .

Proof. Let $\tau_1 = \min\{\ell : \zeta_\ell = 1\}$ and, for $i \geq 1$, let $\tau_{i+1} = \min\{\ell - \tau_i : \ell \geq \tau_i \text{ and } \zeta_\ell = 1\}$. Thus, τ_1 is the index of the first occurrence of state 1 and τ_2, τ_3, \dots are the distances between subsequent occurrences of state 1 in the sequence. Also, let $\tilde{\tau} = \min\{h - \ell : \ell \leq h \text{ and } \tau_\ell = 1\}$, that is, the distance between h and the last occurrence of the state 1 in the sequence $(\zeta_0, \zeta_1, \dots, \zeta_h)$. It is easy to see that

$$\mathbf{P}(N_h = h - k, \tau_1 = t_1, \dots, \tau_k = t_k, \tilde{\tau} = \tilde{t}) = \begin{cases} p^{h-2k} q^k & \text{if } \tilde{t} \geq 1, 0 \leq k \leq \lfloor h/2 \rfloor, \\ p^{h-2k+1} q^k & \text{if } \tilde{t} = 0, 1 \leq k \leq \lfloor (h+1)/2 \rfloor. \end{cases}$$

Thus, adding up over all the possible choices of $t_1, \dots, t_k, \tilde{t}$, having in mind the restrictions $t_1 \geq 1, t_2 \geq 2, \dots, t_k \geq 2$ and $t_1 + \dots + t_k + \tilde{t} = h$; we obtain

$$\begin{aligned} \mathbf{P}(N_h = h - k \text{ and } \zeta_h = 0) &= \binom{h-k}{k} p^{h-2k} q^k & \text{for } 0 \leq k \leq \lfloor h/2 \rfloor \\ \mathbf{P}(N_h = h - k \text{ and } \zeta_h = 1) &= \binom{h-k}{k-1} p^{h-2k+1} q^k & \text{for } 1 \leq k \leq \lfloor (h+1)/2 \rfloor, \end{aligned}$$

therefore,

$$\mathbf{E}[z^{N_h}] = \sum_{k=0}^{\lfloor h/2 \rfloor} \binom{h-k}{k} p^{h-2k} q^k z^{h-k} + \sum_{k=1}^{\lfloor (h+1)/2 \rfloor} \binom{h-k}{k-1} p^{h-2k+1} q^k z^{h-k} \quad (40)$$

Now, for the first term, we have that

$$\sum_{k=0}^{\lfloor h/2 \rfloor} \binom{h-k}{k} p^{h-2k} q^k z^{h-k} = (pz)^h \sum_{k=0}^{\lfloor h/2 \rfloor} \binom{h-k}{k} x^k,$$

where $x = \frac{q}{zp^2}$. By the standard saddle point formula, after noticing that the function

$$\phi(t) = \lim_{h \rightarrow \infty} h^{-1} \ln \left[\binom{h-th}{th} x^{th} \right] = (1-t) H \left(\frac{t}{1-t} \right) + t \ln(x),$$

(where $H(\alpha) = \log \left(\frac{1}{\alpha^\alpha (1-\alpha)^{1-\alpha}} \right)$ stands for natural entropy), reaches its maximum at $t^* = \frac{1}{2}(1 - \epsilon)$, where $\epsilon = 1/\sqrt{1+4x}$ and $\phi''(t^*) = \frac{-4}{\epsilon(1-\epsilon)(1+\epsilon)}$, we have that, as $h \rightarrow \infty$

$$\sum_{k=0}^{\lfloor h/2 \rfloor} \binom{h-k}{k} x^k \sim \sqrt{\frac{(1-t^*)}{t^*(1-2t^*)|\phi''(t^*)|}} \times e^{h\phi(t^*)} = \frac{(1+\epsilon)}{2} \left(\frac{1+\sqrt{1+4x}}{2} \right)^h. \quad (41)$$

For the second term in (40), we have that

$$\sum_{k=1}^{\lfloor (h+1)/2 \rfloor} \binom{h-k}{k-1} p^{h-2k+1} q^k z^{h-k} = p(pz)^h \sum_{k=1}^{\lfloor (h+1)/2 \rfloor} \binom{h-k}{k-1} x^k,$$

Using a similar saddle point estimate, we have that as $h \rightarrow \infty$,

$$\sum_{k=1}^{\lfloor (h+1)/2 \rfloor} \binom{h-k}{k-1} x^k \sim \frac{(1-\epsilon^2)}{4\epsilon} \left(\frac{1+\sqrt{1+4x}}{2} \right)^h. \quad (42)$$

Now, combining the asymptotics (41) and (42) into Eq. (40), the lemma follows. \square

Proof of Lemma 2.15. Notice that, for $\sigma \sim \nu_h$, we have that $\{\sigma(u_i)\}_{i=0}^h$ is a Markov chain with state space $\{0, 1\}$, transition probabilities given by

$$m = \begin{bmatrix} 1/(1+\omega) & \omega/(1+\omega) \\ 1 & 0 \end{bmatrix}$$

and initial distribution

$$\pi = \begin{bmatrix} 1/(1+\omega) & \omega/(1+\omega) \end{bmatrix}.$$

To estimate $\mathbf{E}_{\sigma \sim \mu_{h,\omega}} [\theta^{\#\{i:\zeta_i=0\}}]$ for any $\theta > 0$, we apply Lemma 2.17. For the Markov process $\{\zeta_i\}_{i \geq 0}$ with transition rates $p_{0 \rightarrow 0} = \frac{1}{1+\omega}$, $p_{0 \rightarrow 1} = \frac{\omega}{1+\omega}$, $p_{1 \rightarrow 0} = 1$, $p_{1 \rightarrow 1} = 0$ and $\zeta_0 = 0$, it is the case that $\{\zeta_{i+1}\}_{i=0}^h$ has the same distribution as $\{\sigma(u_i)\}_{i=0}^h$. In particular, it holds that $\mathbf{E}_{\sigma \sim \nu_h} [\theta^{N_{\mathbf{P}}(\sigma)}] = \mathbf{E} [\theta^{N_h}]$.

Now, plugging in the asymptotic from Lemma 2.17 for $\theta = \frac{1.01\omega(1+\omega)}{\lambda}$, we get

$$\mathbf{E}_{\sigma \sim \nu_h} [\theta^{N_{\mathbf{P}}(\sigma)}] = \mathbf{E} [\theta^{N_h}] \leq C_{\delta,b} \left(\frac{1.01\omega}{2\lambda} \left[1 + \sqrt{1 + \frac{4\lambda}{1.01}} \right] \right)^h \leq C_{\delta,b} \left(\frac{1.01\omega}{\lambda^{1/2}} \right)^h. \quad (43)$$

For the last inequality, we used the fact that $1 + \sqrt{1 + 4\lambda/1.01} \leq 2\lambda^{1/2}$ which holds for $\lambda > (101)^2$. In particular, when $\omega = (1+\delta) \ln b/b$ and $b > \max\{b_0(\delta), 1199\}$, we have that $\lambda > (101)^2$ and, therefore, Eq. (43) holds, proving the lemma. \square

2.3.3 Lower bound for a discrete approximation to ν : Proof of part 3 of Theorem 2.10

In order to prove part 3 of Theorem 2.10, we need an approximation of ν through finite trees with discrete boundaries that is robust enough, to handle the analysis that we carried out before to show the slow down in the Glauber dynamics for ν_h .

Notice that, in the region where ν is extremal, the measure is approximable by discrete boundaries [42], but the region in which we are interested is precisely where ν is non-extremal. Instead, we will use the construction exhibited in [97]. This, in particular, fits

the results we obtained in Section 2.2.2.2. Then, we proceed to extend the analysis carried out in the previous section to this case. Let us recall first the result from [97].

Theorem 2.18. [97, Corollary 10] *Given any $\omega < 1$ and $b \geq 1$, there exists a sequence of discrete boundary conditions $\{\Gamma_h\}_{h \geq 1}$ such that for every $\epsilon > 0$, there exists $l(\epsilon)$ (not depending on h , but maybe on ω and b), such that for every $v \in T_{h,b_h}$ at distance $\leq h - l(\epsilon)$ from the root, we have*

$$\left| \frac{\text{Prob}_{\sigma \sim \mu_{h,v}}(\sigma_v = 0)}{1/(1+\omega)} - 1 \right| \leq \epsilon.$$

Here, $\mu_h (= \mu_{T_h, \Gamma_h})$ is the independent sets measure defined over T_h , with boundary condition Γ_h (recall also that $\mu_{h,v}$ is short notation for $\mu_{T_h^{(v)}, \Gamma_h^{(v)}}$). In particular, for every edge e at distance $\leq h - l(\epsilon)$ from the root, we have that

$$m_e = \begin{bmatrix} p & 1-p \\ 1 & 0 \end{bmatrix}, \text{ where } \left| \frac{p}{1/(1+\omega)} - 1 \right| \leq \epsilon.$$

Now, we will consider the parsimonious reconstruction scheme $F_{h,k}$ defined in Section 2.2.2. To bound the average sensitivity, we notice that

$$\begin{aligned} \mathbf{E}_{\sigma \sim \mu_h} [\nabla F_{h,k}(\sigma)] &= \frac{1}{|L_k|} \mathbf{E}_{\sigma \sim \mu_h} \left[\sum_{u \in L_k} \mathbf{I} \left(F_{h,k}(\sigma) = 1 \text{ and } F_{h,k}(\sigma^{(u)}) = 0 \right) \right] \\ &\leq \max_{u \in L_k} \mathbf{P}_{\sigma \sim \mu_h} \left(F_h(\sigma) = 1 \text{ and } F_h(\sigma^{(u)}) = 0 \right). \end{aligned} \quad (44)$$

Then, from Observation 1 on page 43, we obtain the following analogue of Lemma 2.14:

Lemma 2.19. *For every $v \in T_h$ such that $d(v, r) = i$, let*

$$f_{v,h} := \mathbf{P}_{\sigma \sim \mu_{h,v}}(F_{h-i,k-i}(\sigma, v) = 0).$$

Then, for any $u \in L_k$, we have that

$$\mathbf{P}_{\sigma \sim \mu_h}(F_{h,k}(\sigma) = 1 \text{ and } F_{h,k}(\sigma^{(u)}) = 0) \leq \mathbf{E}_{\sigma \sim \mu_h} \left[\prod_{i=0}^{k-1} \left(\prod_{j=1}^{b-1} f_{w_{i,j}} \right)^{(1-\sigma(u_i))} \right],$$

where $\mathbf{p} = u_0, u_1, \dots, u_k$, with $u_0 = r$ and $u_k = u$, is the path joining the root and u and, for every $i = 0, \dots, k$, $w_{i,1}, \dots, w_{i,b-1}$ are the children of u_i not in the path \mathbf{p} .

Proof. The proof proceeds similarly to Lemma 2.14. First, due to Observation 1, we have that

$$\begin{aligned} \mathbf{P}_{\sigma \sim \mu_h}(F_{h,k}(\sigma) = 1 \text{ and } F_{h,k}(\sigma^{(u)}) = 0) \\ = \mathbf{P}_{\sigma \sim \mu_h}(G_{h,k}(\sigma, w_{i,j}) = 0 \text{ for } i \in 0, \dots, k-1, j = 1, \dots, b-1). \end{aligned}$$

Now, we expose the configurations along the path \mathbf{p} . For this purpose, as in Lemma 2.14, given a configuration $\zeta : \{0, \dots, k\} \rightarrow \{0, 1\}$, let $\alpha_\zeta := \mathbf{P}_{\sigma \sim \mu_h}(\sigma(u_i) = \zeta_i \text{ for } i = 0, \dots, k)$. Now,

$$\begin{aligned} \mathbf{P}_{\sigma \sim \mu_h}(F_{h,k}(\sigma) = 1 \text{ and } F_{h,k}(\sigma^{(v)}) = 0) \\ \leq \sum_{\zeta} \alpha_\zeta \prod_{i=0}^{k-1} \prod_{j=1}^{b-1} \mathbf{P}_{\sigma \sim \mu_h}(G_{h,k}(\sigma, w_{i,j}) = 0 : \sigma(u_i) = \zeta_i) \\ \leq \sum_{\zeta} \alpha_\zeta \prod_{\substack{i=0, \dots, k-1 \\ \zeta_i=0}} \prod_{j=1}^{b-1} \mathbf{P}_{\sigma \sim \mu_h}(G_{h,k}(\sigma, w_{i,j}) = 0 : \sigma(u_i) = 0) \\ = \sum_{\zeta} \alpha_\zeta \prod_{\substack{i=0, \dots, k-1 \\ \zeta_i=0}} \prod_{j=1}^{b-1} f_{h,w_{i,j}} \\ \leq \mathbf{E}_{\sigma \sim \mu_h} \left[\prod_{i=0}^{k-1} \left(\prod_{j=1}^{b-1} f_{w_{i,j}} \right)^{1-\sigma(u_i)} \right] \end{aligned}$$

□

Proof of part 3 of Theorem 2.10. In the following, we will assume that $k = k(h) \leq h - l(\epsilon)$, where $\epsilon = \left(\exp\left(\frac{1.01\omega^2 b}{\lambda}\right) - 1 \right)$ (recall this choice from the proof of Theorem 2.8). Now, we can obtain an upper bound on the average sensitivity as follows:

$$\begin{aligned} \mathbf{E}_{\sigma \sim \mu_h} [\nabla F_{h,k}(\sigma)] \\ \leq \max_{u \in L_k} \mathbf{P}_{\sigma \sim \mu_h}(F_{h,k}(\sigma) = 1 \text{ and } F_{h,k}(\sigma^{(v)}) = 0), \quad \text{by Eq. (44)} \\ \leq \max_{u \in L_k} \mathbf{E}_{\sigma \sim \mu_h} \left[\prod_{i=0}^{k-1} \left(\prod_{j=1}^{b-1} f_{w_{i,j}} \right)^{1-\sigma(u_i)} \right], \quad \text{by Lemma 2.19} \\ \leq \max_{u \in L_k} \mathbf{E}_{\sigma \sim \mu_h} \left[\left(\frac{1.01^{1/b}}{1 + \omega} \right)^{(b-1)\#\{i \leq k-1 : \sigma(u_i) = 0\}} \right], \quad \text{by the estimate (1) in Theorem 2.8} \\ \leq \max_{u \in L_k} \mathbf{E}_{\sigma \sim \mu_h} \left[\left(\frac{1.01\omega(1 + \omega)}{\lambda} \right)^{N_{\mathbf{p}}(\sigma)} \right], \quad \text{where } N_{\mathbf{p}}(\sigma) = \#\{i \leq k-1 : \sigma(u_i) = 0\}. \end{aligned}$$

The following lemma (analogue to Lemma 2.15), bounds the above expectation. Its proof is presented afterwards.

Lemma 2.20. *For all $\delta > 0$ and $b > \max\{b_0(\delta), 1219\}$, setting $\omega = (1 + \delta) \ln b / b$, we have that for any $u \in L_k$, where $k \leq h - l(\epsilon)$,*

$$\mathbf{E}_{\sigma \sim \mu_h} \left[\left(\frac{1.01\omega(1 + \omega)}{\lambda} \right)^{N_{\mathbf{P}}(\sigma)} \right] \leq C_{\delta, b} \left(\frac{1.01\omega}{\lambda^{1/2}} \right)^k.$$

Now, due to Theorem 2.4, recalling the non-triviality of the scheme $F_{h,k}$ whenever $\delta > 0$ and $b \geq b_0(\delta)$ from Item 3 of Theorem 2.8, we have that

$$\tau_{\text{rel}} \geq C'_{\delta, b} \left(\frac{\lambda^{1/2}}{1.01\omega} \right)^h \geq C''_{\delta, b} \left(n^{1 + \frac{\ln(\lambda/(1.01\omega b)^2)}{2 \ln b}} \right)$$

Therefore, the conclusion follows (further details are identical to the proof of part 3 of Theorem 2.10 in the previous section). \square

We will proceed now to prove Lemma 2.20, for which, correspondingly, we will require the following lemma.

Lemma 2.21. *Let ζ_0, ζ_1, \dots be a Markov process with state space $\{0, 1\}$, such that $\zeta_0 = 0$ and with transition rates $p_{0 \rightarrow 0} = p$, $p_{0 \rightarrow 1} = q$, $p_{1 \rightarrow 0} = 1$, $p_{1 \rightarrow 1} = 0$. Let $N_h := \#\{1 \leq i \leq h : \zeta_i = 0\}$. Now, let $\bar{\zeta}_0, \bar{\zeta}_1, \dots$ be a ‘perturbed’ version of the chain, in the sense that the transition rate $\bar{p}_{0 \rightarrow 0}$ (and therefore $\bar{p}_{0 \rightarrow 1}$) is now inhomogeneous but such that for some $\epsilon > 0$, $\left| \frac{\bar{p}_{0 \rightarrow 0}^{(i)}}{p} - 1 \right| \leq \epsilon$. Then, if $\bar{N}_h = \#\{1 \leq i \leq h : \bar{\zeta}_i = 0\}$, we have that for any $z > 0$:*

$$\mathbf{E} \left[z^{\bar{N}_h} \right] \leq (1 + \epsilon) \mathbf{E} \left[(z(1 + \epsilon))^{N_h} \right].$$

Proof. Using the same notation as in the proof of Lemma 2.17, we have that

$$\begin{aligned} \mathbf{P} \left[\bar{N}_h = h - k, \tau_1 = t_1, \dots, \tau_k = t_k, \tilde{\tau} = \tilde{t} \right] \\ \leq \begin{cases} (1 + \epsilon)^{h-k} p^{h-2k} q^k & \text{if } \tilde{t} \geq 1, 0 \leq k \leq \lfloor h/2 \rfloor, \\ (1 + \epsilon)^{h-k+1} p^{h-2k+1} q^k & \text{if } \tilde{t} = 0, 1 \leq k \leq \lfloor (h+1)/2 \rfloor. \end{cases} \end{aligned}$$

Therefore,

$$\begin{aligned}\mathbf{P}(\bar{N}_h = h - k \text{ and } \zeta_h = 0) &\leq \binom{h-k}{k} (1 + \epsilon)^{h-k} p^{h-2k} q^k && \text{for } 0 \leq k \leq \lfloor h/2 \rfloor \\ \mathbf{P}(\bar{N}_h = h - k \text{ and } \zeta_h = 1) &\leq \binom{h-k}{k-1} (1 + \epsilon)^{h-k+1} p^{h-2k+1} q^k && \text{for } 1 \leq k \leq \lfloor (h+1)/2 \rfloor\end{aligned}$$

This leads to

$$\begin{aligned}\mathbf{E} \left[z^{\bar{N}_h} \right] &\leq \sum_{k=0}^{\lfloor h/2 \rfloor} \binom{h-k}{k} (1 + \epsilon)^{h-k+1} p^{h-2k} q^k z^{h-k} \\ &\quad + \sum_{k=1}^{\lfloor (h+1)/2 \rfloor} \binom{h-k}{k-1} (1 + \epsilon)^{h-k+1} p^{h-2k+1} q^k z^{h-k} \\ &= (1 + \epsilon) \mathbf{E} \left[(z(1 + \epsilon))^{N_h} \right].\end{aligned}$$

□

Proof of Lemma 2.20. The proof goes along the lines of Lemma 2.15.

For $\sigma \sim \mu_h$, we have that $\{\sigma(u_i)\}_{i=0}^k$ is an inhomogeneous Markov chain with state space $\{0, 1\}$, i^{th} transition matrix given by $m^{(u_i, u_{i+1})}$ and initial distribution

$$\pi = \begin{bmatrix} \mathbf{P}_{\sigma \sim \mu_h}(\sigma(r) = 0) & \mathbf{P}_{\sigma \sim \mu_h}(\sigma(r) = 1) \end{bmatrix}.$$

Now, if we consider the inhomogeneous Markov process $\{\zeta_i\}_{i=0}^k$ such that $\zeta_0 = 0$, with transition matrices $\{m^{(i)}\}_{i \geq 1}$ such that

$$m^{(1)} := \begin{bmatrix} \mathbf{P}_{\sigma \sim \mu_h}(\sigma(r) = 0) & \mathbf{P}_{\sigma \sim \mu_h}(\sigma(r) = 1) \\ 1 & 0 \end{bmatrix}$$

and, for $i \geq 1$,

$$m^{(i+1)} = m^{(u_i, u_{i+1})},$$

then it is clear that $\{\sigma(u_i)\}_{i=0}^k$ has the same distributions as $\{\zeta_{i+1}\}_{i=0}^k$. In particular, $\mathbf{E}_{\sigma \sim \mu_h} [\theta^{N_{\mathbf{P}}(\sigma)}] = \mathbf{E} [\theta^{\bar{N}_k}]$. Furthermore, the conditions of Lemma 2.21 hold with $\epsilon = \left(\exp \left(\frac{1.01 \omega^2 b}{\lambda} \right) - 1 \right)$.

Therefore, using the comparison in Lemma 2.21 and the asymptotic in Lemma 2.17, for

$\theta = \frac{1.01\omega(1+\omega)}{\lambda}$ and $k = k(h) := h - l(\epsilon)$, we get that

$$\begin{aligned} & \mathbf{E}_{\sigma \sim \mu_h} \left[\theta^{N_{\mathbf{P}}(\sigma)} \right] \\ &= \mathbf{E} \left[\theta^{\bar{N}_{h-l(\epsilon)}} \right] \\ &\leq C_{\delta,b} \left(\frac{1.01\omega}{2\lambda} \exp \left(\frac{1.01\omega^2 b}{\lambda} \right) \left[1 + \sqrt{1 + \frac{4\lambda}{1.01} \exp \left(-\frac{1.01\omega^2 b}{\lambda} \right)} \right] \right)^h. \end{aligned}$$

Finally, we use the inequality

$$1 + \sqrt{1 + \frac{4\lambda}{1.01} \exp \left(-\frac{1.01\omega^2 b}{\lambda} \right)} \leq 2\lambda^{1/2} \exp \left(-\frac{1.01\omega^2 b}{\lambda} \right),$$

which holds whenever $\lambda \geq (101)^2$. In particular, when $\omega = (1 + \delta) \ln b/b$ and $b > \max \{b_0(\delta), 1219\}$, we have that $\lambda \geq (101)^2$, finalizing the proof of the lemma. \square

CHAPTER III

LATTICE GEOMETRIES

In this chapter, we will focus on the well-studied case of the square lattice \mathbb{Z}^2 and provide a new lower bound for the uniqueness threshold for the independent sets model. Our technique refines and builds on the tree of self-avoiding walks approach of Weitz, resulting in a new technical sufficient criterion (of wider applicability) for establishing strong spatial mixing (and hence uniqueness) for the independent sets model. Our new criterion achieves better bounds on strong spatial mixing when the graph has extra structure, improving upon what can be achieved by just using the maximum degree. Applying our technique to \mathbb{Z}^2 , we prove that strong spatial mixing holds for all $\lambda < 2.3882$, improving upon the work of Weitz that held for $\lambda < 27/16 = 1.6875$. Our results imply a fully-polynomial deterministic approximation algorithm for estimating the partition function, as well as rapid mixing of the associated Glauber dynamics to sample from the independent sets distribution for $\lambda < 2.3882$.

The study of interacting particle systems on lattice graphs is one of the cornerstones of statistical physics. In this area, the interrelation between physical intuition and mathematical rigorous results, has led to stimulating insights through the years. For instance, two notable results that build on such a relation are Lars Onsager's determination of the exact threshold of spontaneous magnetization for the Ising model in \mathbb{Z}^2 [84] and the exact solvability of the hard-core model in the hexagonal lattice by Richard Baxter [8]. We refer the reader also to the excellent books [9, 76] and the surveys regarding Glauber dynamics for interacting particle systems [58, 73].

This chapter is divided into four sections. In Section 3.1, we describe the so called path-tree decomposition of a graph. This is an object that allows the translation of certain information such as marginals and influence decay from the context of general graphs to

the framework of trees. In Section 3.2, we describe a contraction condition that implies a strong form of influence decay called strong spatial mixing. We present such condition, in particular, for a class of structured trees called *branching trees*. In Section 3.3, we use this machinery to extend the domain where strong spatial mixing is known to hold for the independent sets model in \mathbb{Z}^2 . In particular, we introduce a condition that we call DMS, that implies contraction (and therefore strong spatial mixing) for the independent sets model on branching trees. A corresponding condition exists for other spin systems (for instance, the well known Ising model) that allows us to establish similar results. This is presented in Section 3.4.

3.1 The path tree

Let us consider a fixed graph $G = (V, E)$. We denote by \mathcal{P}_G the set of all possible paths on the graph and by $\mathcal{P}_{G,v}$ the paths that ‘start’ at the vertex v . Given a subset $X \subseteq \mathcal{P}_G$, we define an induced graph structure over X , $G_X = (V_X, E_X)$, where $V_X = X$ and, given two paths $p, p' \in X$, we say that $p \sim p'$ iff there exists $x_0, \dots, x_{k+1} \in V$ such that $p = (x_0, \dots, x_k)$ and $p' = (x_0, \dots, x_k, x_{k+1})$. We call such relation *augmentation*. Notice that the graph structure of G_X is that of a forest. Moreover, if $X \subseteq \mathcal{P}_{G,v}$, G_X has, instead, a tree structure.

We say that a path $p = x_0, \dots, x_k$ is of type χ_i (i.e., $p \in \chi_i$), if it does not contain a cycle of length $\leq i$. A path $p = x_0, \dots, x_k$ is *simple* or, of type χ_∞ (i.e. $p \in \chi_\infty$), if it does not contain a cycle (of any length).

Definition 3.1. Given $v \in V$, and $i \geq 1$ (or $i = \infty$), we define $\mathcal{P}_{G,v}^{(i)}$ as the set of paths $p = x_0, \dots, x_k, x_{k+1}$ such that

1. $x_0 = v$.
2. Either p or the reduced path $p' := x_0, \dots, x_{k-1}$ is of type χ_i .

It is clear, from the definition, that such sets form a decreasing structure, that is,

$$\mathcal{P}_{G,v} \supseteq \mathcal{P}_{G,v}^{(1)} \supseteq \mathcal{P}_{G,v}^{(2)} \supseteq \dots \supseteq \mathcal{P}_{G,v}^{(\infty)}. \quad (45)$$

Also, notice that, for a finite graph G , if i_0 is the length of the longest cycle in G , then $\mathcal{P}_{G,v}^{(i)}$ is infinite for $i < i_0$, and finite for $i \geq i_0$. In particular, $\mathcal{P}_{G,v}^{(\infty)}$ is finite.

We say that a path $p \in \mathcal{P}_{G,v}^{(i)}$ is *terminal* if $p \notin \chi_i$. Notice that a terminal path p can be uniquely decomposed as the concatenation of paths p_{ini} and p_{fin} , where $p_{\text{ini}} \in \chi_i$ and p_{fin} is a cycle. Following, we define the concept of *orientation* for terminal paths in $\mathcal{P}_{G,v}^{(i)}$.

Definition 3.2. For every $v \in G$, set a total order $<_v$ among the neighbours of v . The cycle $p = (v_0, v_1, \dots, v_{k-1}, v_k)$, where $v_0 = v_k = v$, is positively oriented if $v_1 >_v v_{k-1}$. Otherwise, p is negatively oriented. Given a terminal path $p \in \mathcal{P}_{G,v}^{(i)}$, we say that p is positively (negatively) oriented if p_{fin} is positively (negatively) oriented.

The graph $\mathcal{P}_{G,v}^{(\infty)}$ is also called *path-tree decomposition* [44] or *tree of self-avoiding walks* [114] of G at the vertex v . Other references in which a similar construction is used are [11, 39, 83]. The relevance of the path-tree decomposition of the graph relies in the fact that it translates the calculation of marginals in the graph to the calculation of marginals in the path-tree. This observation is particularly useful to derive spatial properties of the spin system defined on the graph from corresponding spatial properties for trees.

From now on, we will restrict ourselves to 2-spin systems.

Definition 3.3. Let T be the induced graph structure on $\mathcal{P}_{G,v}^{(i)}$. Given a specification $\gamma = (\psi, \phi)$ over G , we define the lifted specification $\gamma' = (\psi', \phi')$ over T in the following way:

- For every $p \in \mathcal{P}_{G,v}^{(i)}$ such that p is not terminal,

$$\psi'_p := \psi_v,$$

where v is the ending vertex of p .

- For every $p \in \mathcal{P}_{G,v}^{(i)}$ such that p is terminal,

$$\psi'_p(x) := \begin{cases} \mathbf{I}(x=0) & \text{if } p \text{ is negatively oriented} \\ \mathbf{I}(x=1) & \text{if } p \text{ is positively oriented} \end{cases}. \quad (46)$$

- For every $p, q \in \mathcal{P}_{G,v}^{(i)}$ such that $p \sim q$,

$$\phi'_{p,q} := \phi_{v,w},$$

where v and w are the ending vertices of p and q , respectively.

Let us denote by μ , the Gibbs' measure associated with the specification γ in G . If T is finite, we denote by μ_T the Gibbs' measure associated with γ' on T . This is called the lifting of μ in T . Also, let us denote by μ_{T_h} the Gibbs' measure associated with the specification γ' on the restricted tree T_h .

The following relation between the measures μ and $\mu_{(\infty)}$ is the key to our approach.

Theorem 3.4 (SAW Tree Representation, Theorem 3.1 in [114]). *If T is the path-tree of the graph G at v . Then,*

$$\mathbf{P}_{\sigma \sim \mu}(\sigma(v) = x) = \mathbf{P}_{\sigma \sim \mu_T}(\sigma(r) = x).$$

In the case of multi-spin systems, unfortunately, such lifting does not exist. However, a tree recurrence with similar applicability holds. For instance, see [39, 83].

3.2 Strong spatial mixing

Now, we will proceed to define the concept of strong spatial mixing (SSM). This is a standard concept that can also be found, for instance, in [73]. We extend the definition slightly, to include what we call *generalized assignments*, leading the definition of *generalized strong spatial mixing* (gSSM). This extended notion will be particularly useful to show mixing properties in a tree by showing mixing properties in a, probably simpler, supertree.

A *partial assignment* is a function $\rho : A \rightarrow \{0, 1\}$, where $A \subseteq G$. If $\mathbf{P}_{\sigma \sim \mu_G}(\sigma_A = \rho) > 0$, we define the measure $\mu_{G, \rho}$ such that

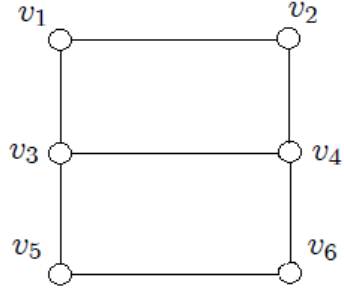
$$\mathbf{P}_{\sigma \sim \mu_{G, \rho}}(\sigma = \eta) = \mathbf{P}_{\sigma \sim \mu_G}(\sigma = \eta : \sigma_A = \rho), \quad \eta \in \Omega_G.$$

If ρ and ρ' are partial assignments on the graph G , we define

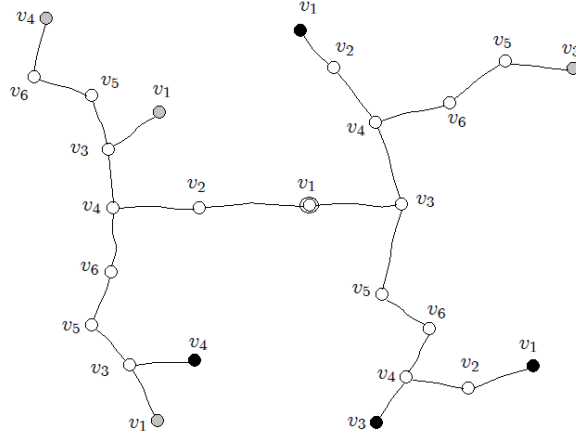
$$\text{ssm}_{G, v}(\rho, \rho') := \left| \mathbf{P}_{\sigma \sim \mu_{G, \rho}}(\sigma(v) = 0) - \mathbf{P}_{\sigma \sim \mu_{G, \rho'}}(\sigma(v) = 0) \right|.$$

Let $\mathcal{B}_k(v)$ be the set of pairs of assignments ρ and ρ' , with the same support, that differ only at vertices at distance $\geq k$ from v . We define the rate of *strong spatial mixing* (ssm), as follows:

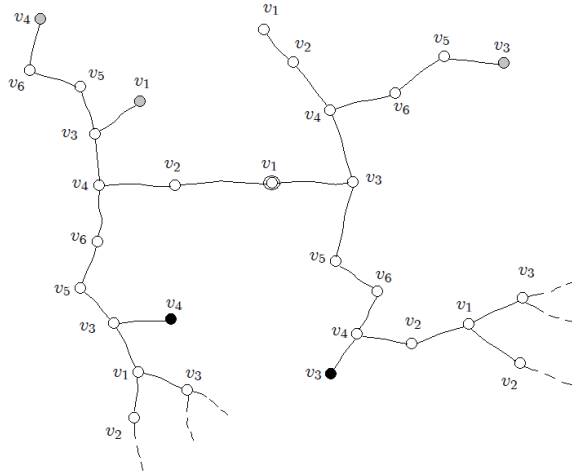
$$\text{ssm}_{G, v}^{(k)} := \sup_{(\rho, \rho') \in \mathcal{B}_k(v)} \text{ssm}_v(\rho, \rho').$$



(a) Graph G



(b) $\mathcal{P}_{G,v_1}^{(\infty)}$



(c) $\mathcal{P}_{G,v_1}^{(4)}$

Figure 3: Path-tree of the graph G . Each vertex v in the tree is a path $v = w_1, w_2, \dots, w_k$ in the graph. We label each vertex $v = w_1, w_2, \dots, w_k$ in the tree with its terminal vertex w_k . The positively oriented terminal paths are colored in gray, while the negatively oriented in black. We have fixed the order $v_1 > v_2 > \dots > v_6$

In the case of trees, we set v to be the root, and define

$$\text{ssm}_T^{(k)} := \text{ssm}_{T,r}^{(k)} = \sup_{(\rho, \rho') \in \mathcal{B}_k(r)} \text{ssm}_{T,r}(\rho, \rho').$$

Now, for a family of graphs $\{G_n\}_{n \geq 1}$, we say that *strong spatial mixing* (SSM) holds if there exist $C > 0$ and $\gamma \in (0, 1)$ such that for all $n \geq 1$, all $k \geq 1$ and all $v \in G_n$,

$$\text{ssm}_{G_n, v}^{(k)} \leq C\gamma^k. \quad (47)$$

Similarly, for a family of trees $\{T_h\}_{h \geq 1}$, we say that strong spatial mixing holds, if there exist $C > 0$ and $\gamma \in (0, 1)$ such that for all $h, k \geq 1$,

$$\text{ssm}_{T_h}^{(k)} \leq C\gamma^k. \quad (48)$$

In the case of trees, we extend the concept of strong spatial mixing to include a bigger class of assignments. A *generalized assignment* on the tree T is a function $\Gamma : A \rightarrow [0, 1]$, where $A \subseteq T$. To define the measure $\mu_{T, \Gamma}$, let $T^{(\Gamma)}$ be the tree resulting from deleting the subtrees subtended at the children of vertices $v \in A$. Now, at the vertices $v \in A \cap T^{(\Gamma)}$ assign the self-interactions

$$\psi_v^{(\Gamma)}(0) = \Gamma(v), \psi_v^{(\Gamma)}(1) = 1 - \Gamma(v).$$

Then, $\mu_{T, \Gamma}$ will be the Gibbs' measure associated with the resulting tree $T^{(\Gamma)}$ and the modified specification.

Notice that $\mu_{T, \Gamma}$ and $\mu_{T, \rho}$ coincide when Γ is a *discrete assignment*, that is, if Γ takes only the values $\{0, 1\}$. Now, we define the *rate of generalized strong spatial mixing* (gssm) as follows:

$$\text{gssm}_T(\Gamma, \Gamma') := \left| \mathbf{P}_{\sigma \sim \mu_{T, \Gamma}}(\sigma(r) = 0) - \mathbf{P}_{\sigma \sim \mu_{T, \Gamma'}}(\sigma(r) = 0) \right|.$$

Let $\bar{\mathcal{B}}_k$ be the pair of generalized assignments Γ and Γ' , with the same support, that differ only at vertices at distance $\geq k$ from the root. We define

$$\text{gssm}_T^{(k)} := \sup_{(\Gamma, \Gamma') \in \bar{\mathcal{B}}_k} \text{gssm}(\Gamma, \Gamma').$$

In particular, we say that gSSM holds for the family of trees $\{T_h\}_{h \geq 1}$, if there exist $C > 0$ and $\gamma \in (0, 1)$ such that for all $h, k \geq 1$,

$$\text{gssm}_{T_h}^{(k)} \leq C\gamma^k$$

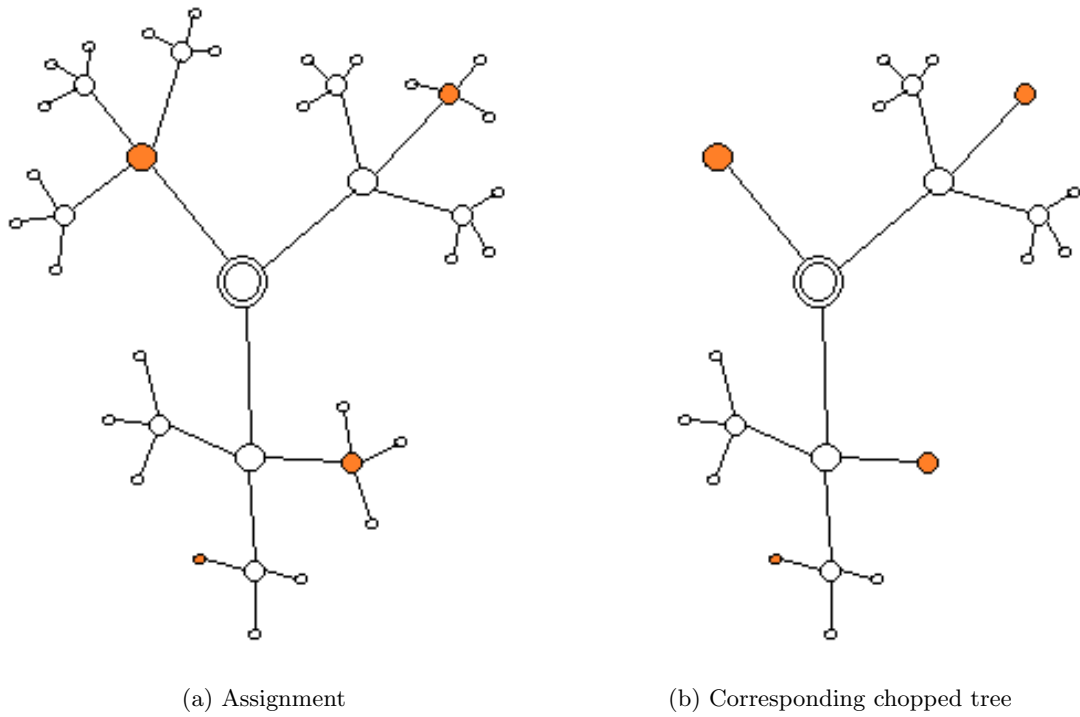


Figure 4: Given an assignment, the spin at the root of the tree ‘ignores’ the vertices below the support of the assignment. This fact motivates the definition of generalized assignment on trees.

It is clear from the definition that

$$\text{ssm}_T^{(k)} \leq \text{gssm}_T^{(k)}. \quad (49)$$

Therefore, gSSM for a family of trees implies SSM for the family.

The definition of $\text{gssm}_T^{(k)}$ can be further simplified:

Proposition 3.5. *Let $\tilde{\mathcal{B}}_k$ be the set consisting of the pairs of boundary conditions Γ, Γ' defined on $L_k \cup A$, where $A \subseteq L_{\leq k}$, and such that $\Gamma_A = \Gamma'_A$. Then, it is the case that*

$$\text{gssm}_T^{(k)} := \sup_{(\Gamma, \Gamma') \in \tilde{\mathcal{B}}_k} \text{ssm}_v(\Gamma, \Gamma')$$

Proof. Given a generalized assignment Γ , define the corresponding assignment $\bar{\Gamma}$ on $\text{supp } \Gamma \cap T^{(\Gamma)}$ by

$$\bar{\Gamma}(v) = \begin{cases} \mathbf{P}_{\sigma \sim \mu_{T(v), \Gamma(v)}}(\sigma(v) = 0) & \text{if } v \in L_k \cap T^{(\Gamma)} \\ \Gamma(v) & \text{if } d(v, r) < k \end{cases}.$$

Then, by the tree-recursion Eq. (14), we have that

$$\mu_{T, \Gamma} = \mu_{T, \bar{\Gamma}}.$$

The proposition follows immediately from this fact. \square

The following proposition connects the concept of spatial mixing for graphs and trees.

Proposition 3.6. *Let T be the path-tree of the graph G . Let μ_G be the Gibbs' measure associated with the given specification on the graph G , and let μ_T be the corresponding lifted measure defined on T (Definition 3.3). Then, for all $k \geq 1$,*

$$\text{ssm}_{\mu_G}^{(k)}(v) \leq \text{ssm}_{\mu_T}^{(k)} \leq \text{gssm}_{\mu_T}^{(k)}.$$

Proof. Given an assignment $\rho : A \rightarrow \{0, 1\}$, where $A \subseteq G$, let $\tilde{\rho} : \tilde{A} \rightarrow \{0, 1\}$ be the assignment defined on T , where \tilde{A} is the set of paths p whose ending vertex v belongs to A , in which case $\tilde{\rho}(p) := \rho(v)$. From Definition 3.3 is clear that the measure $\mu_{T, \tilde{\rho}}$ is the lifted measure of $\mu_{G, \rho}$. Therefore, due to Theorem 3.4,

$$\mathbf{P}_{\sigma \sim \mu_{G, \rho}}(\sigma(v) = 0) = \mathbf{P}_{\sigma \sim \mu_{T, \tilde{\rho}}}(\sigma(r) = 0). \quad (50)$$

From this fact, the first inequality of the proposition clearly follows. The second inequality is a restatement of Eq. (49). \square

Notice that the reverse inequality does not hold since there are assignments in \mathbf{T} that do not result from mapping a boundary condition from G . The Ising model in \mathbb{Z}^2 , as we will see in Section 3.4, gives an example of this situation.

3.2.1 Contraction principle for the regular tree

Let us consider a 2-spin system defined over the finite tree \mathbf{T} . Given $v \in \mathbf{T}$, let

$$\alpha_v := \mathbf{P}_{\sigma \sim \mu_v}(\sigma(v) = 0)$$

(recall that μ_v is short for $\mu_{\mathbf{T}(v)}$). Now, due to the recurrence for trees stated in Eq. (15), we have that, if v has children w_1, \dots, w_b , then

$$\alpha_v := F\left(\left(\prod_{i=1}^b h(\alpha_{w_i})\right)^{1/b}\right), \quad (51)$$

where $F(z) := \frac{1}{1+\mathcal{J}z^b}$, $\mathcal{J} := \psi(1)/\psi(0)$ and, for $z \in [0, 1]$,

$$h(z) := \frac{(\phi(1, 0) - \phi(1, 1))z + \phi(1, 1)}{(\phi(0, 0) - \phi(0, 1))z + \phi(0, 1)}. \quad (52)$$

We will assume that $\phi(1, 0), \phi(0, 1) > 0$ and, w.l.g., $\phi(0, 0) > 0$ ¹. It is easy to notice that under these assumptions, $h \in C^1([0, 1])$. Therefore, in particular, h is a bounded function with bounded derivative

$$h'(z) = \frac{-\det \phi}{[(\phi(0, 0) - \phi(0, 1))z + \phi(0, 1)]^2}.$$

Let us define also,

$$H(z) := \frac{\partial}{\partial z} \log h(z). \quad (53)$$

A *statistic* of the univariate parameter $z \in [a, b]$ is an increasing function $\varphi : [a, b] \rightarrow \mathbb{R}$. Given a statistic $\varphi : [0, 1] \rightarrow \mathbb{R}$, let $m_v := \varphi(\alpha_v)$. In order to show decay of influence of the boundary condition, a common strategy is to prove some form of contraction for the

¹These are natural assumptions to avoid trivial cases.

‘one-step’ iteration given in Eq. (51). More generally, we can prove such contraction for an appropriate statistic φ of the parameter α_v . That is the purpose of Proposition 3.7.

Let us define the intervals $J := h([0, 1])$, and $I := F(J)$. Given a statistic φ , define

$$H^{(\varphi)}(z) := \sup \left\{ \sum_{i=1}^b |H(x_i)| / \varphi'(x_i) : \left(\prod_{i=1}^b h(x_i) \right)^{1/b} = z, x_1, \dots, x_b \in I \right\}. \quad (54)$$

Proposition 3.7. *Let us consider a 2-spin system defined over the b -ary tree T_h of height h . Suppose that the statistic $\varphi : I \rightarrow \mathbb{R}$ satisfies the following conditions,*

1. φ' is bounded away from 0 and ∞ on I . That is,

$$\inf_{z \in I} \varphi'(z) > 0, \quad \sup_{z \in I} \varphi'(z) < \infty.$$

2. There exists $\gamma \in (0, 1)$ such that, for all $z \in I$,

$$F(z)(1 - F(z))\varphi'(F(z))H^{(\varphi)}(z) < \gamma,$$

Then, there exists $C > 0$ such that for every $k, h \geq 1$,

$$gssm_{T_h}^{(k)} \leq C\gamma^k.$$

Proof. Let Γ and Γ' be generalized assignments defined over $L_k \cup A$, and such that $\Gamma_A = \Gamma'_A$.

For $\varepsilon \in [0, 1]$, let Γ_ε be the boundary condition such that

$$\psi_v^{(\Gamma_\varepsilon)}(x) = (1 - \varepsilon)\psi_v^{(\Gamma)}(x) + \varepsilon\psi_v^{(\Gamma')}(x), \quad x \in X, v \in L_k \cup A.$$

In particular we have that, for $v \in A$ and $x \in X$,

$$\psi_v^{(\Gamma_\varepsilon)}(x) = \psi_v^{(\Gamma)}(x) = \psi_v^{(\Gamma')}(x).$$

and

$$\Gamma_0 = \Gamma, \quad \Gamma_1 = \Gamma'.$$

Let us denote by μ_ε the measure compatible with the specification over T and the assignment Γ_ε . Also, given $v \in T$, let $\mu_{\varepsilon, v}$ be the measure compatible with the specification over $T^{(v)}$ and the restriction of the assignment Γ_ε to $T^{(v)}$. Also, let

$$\alpha_v^{(\varepsilon)} := \mathbf{P}_{\sigma \sim \mu_{\varepsilon, v}}(\sigma(v) = 0) \quad \text{and} \quad m_v^{(\varepsilon)} := \varphi\left(\alpha_v^{(\varepsilon)}\right).$$

Now, our objective is to prove that, for every $v \in T$ such that $d(v, r) = i$, we have that

$$\left| \frac{\partial}{\partial \varepsilon} m_v^{(\varepsilon)} \right| \leq C' \gamma^{k-i}, \quad (55)$$

for some constant C' to be determined in a moment. First, notice that if $v \in L_{k-1} \cap A^c$ has children w_1, \dots, w_b , then

$$\alpha_v^{(\varepsilon)} = 1 / \left(1 + \mathcal{J} \prod_{i=1}^b h \left(\alpha_{w_i}^{(\varepsilon)} \right) \right),$$

where

$$\alpha_{w_i}^{(\varepsilon)} = (1 - \varepsilon) \psi_{w_i}^{(\Gamma)}(0) + \varepsilon \psi_{w_i}^{(\Gamma')}(0).$$

Therefore,

$$\frac{\partial}{\partial \varepsilon} m_v^{(\varepsilon)} = -\mathcal{J} \left(\alpha_v^{(\varepsilon)} \right)^2 \varphi' \left(\alpha_v^{(\varepsilon)} \right) \sum_i \left(\prod_{j \neq i} h \left(\alpha_{w_j}^{(\varepsilon)} \right) \right) h' \left(\alpha_{w_i}^{(\varepsilon)} \right) \left(\psi_{w_i}^{(\Gamma')}(0) - \psi_{w_i}^{(\Gamma)}(0) \right).$$

From the boundedness of φ' in I , and the boundedness of h and h' in $[0, 1]$, there exists a constant $C'' > 0$ such that for every $v \in L_{k-1} \cap A^c$,

$$\left| \frac{\partial}{\partial \varepsilon} m_v^{(\varepsilon)} \right| \leq C''.$$

Now, in Eq. (55), we take $C' := C''/\gamma$. Under this choice, for $v \in L_{k-1} \cap A^c$, Eq. (55) trivially holds. On the other hand, if $v \in L_{k-1} \cap A$, clearly $\frac{\partial}{\partial \varepsilon} m_v^{(\varepsilon)} = 0$, and therefore Eq. (55) holds. Now, for $v \in L_i \cap A^c$, with children w_1, \dots, w_b , we have that

$$\alpha_v^{(\varepsilon)} = 1 / \left(1 + \mathcal{J} \prod_{i=1}^b h \left(\alpha_{w_i}^{(\varepsilon)} \right) \right)$$

and, therefore,

$$\begin{aligned} \left| \frac{\partial}{\partial \varepsilon} m_v^{(\varepsilon)} \right| &= \left| \alpha_v^{(\varepsilon)} \left(1 - \alpha_v^{(\varepsilon)} \right) \varphi' \left(\alpha_v^{(\varepsilon)} \right) \sum_{i=1}^b \frac{H \left(\alpha_{w_i}^{(\varepsilon)} \right)}{\varphi' \left(\alpha_{w_i}^{(\varepsilon)} \right)} \frac{\partial}{\partial \varepsilon} m_{w_i}^{(\varepsilon)} \right| \\ &\leq \sup_{z \in I} F(z) (1 - F(z)) \varphi' (F(z)) H^{(\varphi)}(z) \max_{i \in \{1, \dots, b\}} \left| \frac{\partial}{\partial \varepsilon} m_{w_i}^{(\varepsilon)} \right| \\ &\leq \gamma \max_{i \in \{1, \dots, b\}} \left| \frac{\partial}{\partial \varepsilon} m_{w_i}^{(\varepsilon)} \right|, \text{ from condition 2.} \end{aligned}$$

Therefore, assuming that Eq. (55) holds for vertices in L_{i+1} , we get

$$\left| \frac{\partial}{\partial \varepsilon} m_v^{(\varepsilon)} \right| \leq \gamma \times C' \gamma^{h-(i+1)} = C' \gamma^{h-i}.$$

Notice also that, if $v \in L_i \cap A$, trivially $\left| \frac{\partial}{\partial \varepsilon} m_v^{(\varepsilon)} \right| = 0 \leq C' \gamma^{h-i}$. Consequently, by induction, for all $v \in T$, Eq. (55) holds. In particular, if r is the root of the tree,

$$\left| \frac{\partial}{\partial \varepsilon} m_r^{(\varepsilon)} \right| \leq C' \gamma^h.$$

Now,

$$\begin{aligned} & \left| \mathbf{P}_{\sigma \sim \mu_{T,\Gamma}} (\sigma(r) = 0) - \mathbf{P}_{\sigma \sim \mu_{T,\Gamma'}} (\sigma(r) = 0) \right| \\ &= \left| \alpha_r^{(0)} - \alpha_r^{(1)} \right| \\ &\leq \int_0^1 \left| \frac{\partial}{\partial \varepsilon} \alpha_r^{(\varepsilon)} \right| d\varepsilon \\ &= \int_0^1 \left| \left(\frac{\partial}{\partial \varepsilon} m_r^{(\varepsilon)} \right) / \varphi' \left(\alpha_r^{(\varepsilon)} \right) \right| d\varepsilon, \\ &\leq \frac{1}{\inf_{z \in I} \varphi'(z)} \int_0^1 \left| \frac{\partial}{\partial \varepsilon} m_r^{(\varepsilon)} \right| d\varepsilon, \\ &\leq C \gamma^h, \text{ taking } C = C' / \inf_{z \in I} \varphi'(z). \end{aligned}$$

□

Now, we will introduce a condition under which the gSSM condition is monotone respect to the subtree relation.

Definition 3.8. *We say that the spin system is well oriented if it is the case that for some $z^* \in [0, 1]$, $h(z^*) = 1$.*

Proposition 3.9. *If the system is well oriented, then, for every subtree T' of T , we have that*

$$gssm_{T'}^{(k)} \leq gssm_T^{(k)}.$$

Proof. If $\mu_{T'}$ is the measure defined over T' , notice that $\mu_{T'} = \mu_{T,\Gamma}$ where Γ is the generalized assignment such that for all $v \in T/T'$,

$$\Gamma_v(0) = z^*, \Gamma_v(1) = 1 - z^*.$$

The proposition follows. □

Corollary 3.10. *If the system is well oriented and the conditions of Proposition 3.7 hold, then there exists a constant C such that for all $k \geq 1$ and all **subtrees** T' of the regular tree T ,*

$$gssm_{T'}^{(k)} \leq C\gamma^k.$$

Remark 3.11. *An example in which there is no such monotonicity under the subtree relation is given by the specification $\psi(0) = 1$, $\psi(1) = \bar{\lambda}$. $\phi(0,1) = \phi(1,0) = \epsilon$, $\phi(1,1) = 0$, $\phi(0,0) = 1$. Let T be the b -ary tree of height h . In this case, we have that (preserving the notation from the proof of Proposition ??), for $v \in T_h$, with children w_1, \dots, w_b ,*

$$\alpha_v = \frac{1}{1 + \prod_{i=1}^b \left(\frac{\epsilon \alpha_{w_i}}{(1-\epsilon)\alpha_{w_i} + \epsilon} \right)}.$$

Now, letting $m_v := \frac{\alpha_v}{(1-\epsilon)\alpha_v + \epsilon}$, we get

$$m_v = \frac{1}{1 + \bar{\lambda}\epsilon^{b+1} \prod_{i=1}^b m_{w_i}}.$$

Therefore, the observable m_v for this spin system is equivalent to the observable α_v for the independent sets model with fugacity $\lambda = \bar{\lambda}\epsilon^{b+1}$. It is well known [56, 114] that SSM holds for the independent sets model with fugacity λ in the b -ary tree iff $\lambda < \frac{b^b}{(b-1)^{b+1}}$. Therefore, there is SSM in the spin system iff

$$\bar{\lambda}\epsilon^{b+1} < \frac{b^b}{(b-1)^{b+1}}.$$

This property is clearly nonmonotone in b . For instance, if $\epsilon = 1/2$ and $\bar{\lambda} = 30$ there is SSM in the b -ary tree for $b \geq 4$, but not for $b = 3$.

3.2.2 Contraction principle for branching trees

Now, it is our intention to present a contraction condition similar to Proposition 3.7, for a more general class of trees. Given a $t \times t$ matrix M with nonnegative integer entries (that we call *branching matrix*), we define the *branching family* of trees generated by M , \mathcal{F}_M as follows:

- The tree with a single vertex is an element of \mathcal{F}_M of type ℓ , for any $\ell \in \{1, \dots, t\}$.

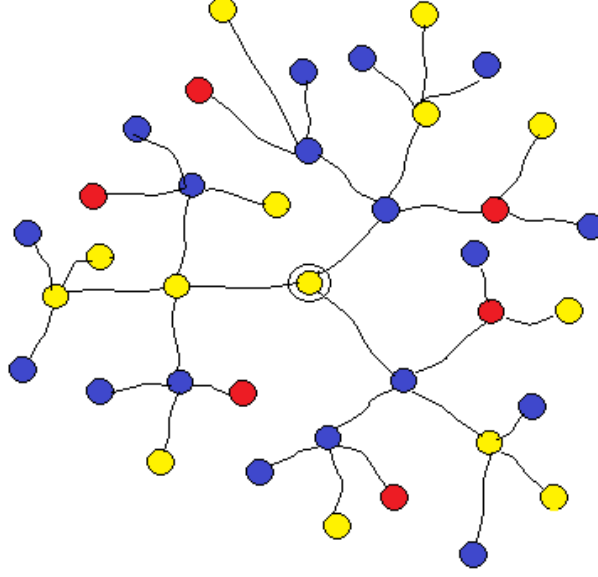


Figure 5: Tree of type 1 in the family \mathcal{F}_M , where $M = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. The yellow vertices correspond to type 1, the blue vertices to type 2 and the red vertices to type 3.

- For every $\ell \in \{1, \dots, t\}$, the tree formed by adjoining $M_{\ell,j}$ trees of type j in \mathcal{F}_M , for $j \in \{1, \dots, t\}$, with a single vertex, is a tree in \mathcal{F}_M of type ℓ .

Moreover, we say that an infinite tree T belongs to \mathcal{F}_M , if for every $h \geq 1$, $T_h \in \mathcal{F}_M$.

We define the family $\mathcal{F}_{\leq M}$, to be the completion of \mathcal{F}_M under the ‘subtree’ relation. That is, the collection of trees T' , such that T' is subtree of some tree $T \in \mathcal{F}_M$. If T is of type ℓ , then we say that T' is of type ℓ in $\mathcal{F}_{\leq M}$.

Example 3.12. The family $\mathcal{F}_{\leq M}$ with $M = [b]$ describes the family of trees with maximum branching b . On the other hand, $\mathcal{F}_{\leq M}$ with $M = \begin{bmatrix} 0 & b-1 \\ 0 & b \end{bmatrix}$ describes the family of trees of maximum degree b .

We say that the branching matrices M and N are *equivalent* if $\mathcal{F}_M = \mathcal{F}_N$. Given a branching matrix M , an alternative way to describe its branching mechanism is by describing its *offspring*. For every $\ell \in \{1, \dots, t\}$, the offspring τ_ℓ is a sequence $\tau_\ell(1), \dots, \tau_\ell(b_\ell)$ with $b_\ell := \sum_{j=1}^t M_{\ell,j}$, such that $M_{\ell,j} = \#\{j' : \tau_\ell(j') = j\}$ for every $j = 1, \dots, t$.

For a $t \times t$ branching matrix M , we consider a set of t statistic $\varphi_1, \dots, \varphi_t$. For the simpler case, when $M = [b]$, and so $t = 1$, we have a single statistic φ (as in the previous section). Our aim is proving contraction for an appropriate set of statistic in this setting, that refines the contraction principle exhibited in Proposition 3.7.

As in the previous section, let us consider a 2-spin system defined over the finite tree T . Given $v \in T$, let $\alpha_v := \mathbf{P}_{\sigma \sim \mu_v}(\sigma(v) = 0)$. Then, if the tree $T^{(v)}$ is of type ℓ , and w_1, \dots, w_{b_ℓ} are the children of v ,

$$\alpha_v := F_\ell \left(\left(\prod_{i=1}^{b_\ell} h(\alpha_{w_i}) \right)^{1/b_\ell} \right),$$

where $F_\ell(z) := \frac{1}{1 + \mathcal{J} z^{b_\ell}}$. J, \mathcal{J}, h and H are defined as in the previous section. In this case, we define

$$I := \bigcup_{\ell=1}^t F_\ell(J).$$

Also, for given statistic $\{\varphi_\ell\}_{\ell=1}^t$, and positive constants $\{\kappa_\ell\}_{\ell=1}^t$, we define

$$\begin{aligned} & H_\ell^{(\varphi, \kappa)}(z) \\ &:= \frac{1}{\kappa_\ell} \sup \left\{ \sum_{j=1}^{b_\ell} \kappa_{\tau_\ell(j)} |H(x_j)| / \varphi'_{\tau_\ell(j)}(x_j) : \left(\prod_{j=1}^{b_\ell} h(x_j) \right)^{1/b_\ell} = z, x_1, \dots, x_{b_\ell} \in I \right\}. \end{aligned}$$

Proposition 3.13. *Suppose that the statistic $\{\varphi_\ell\}_{\ell=1}^t$, $\varphi_\ell : I \rightarrow \mathbb{R}$, satisfy the following:*

1. *For every $\ell \in \{1, \dots, t\}$,*

$$\inf_{z \in I} \varphi'_\ell > 0 \text{ and } \sup_{z \in I} \varphi'_\ell < \infty.$$

2. *There exist $\gamma \in (0, 1)$ and constants $\{\kappa_\ell\}_{\ell=1}^t$ such that for every $\ell \in \{1, \dots, t\}$,*

$$F_\ell(z) (1 - F_\ell(z)) \varphi'_\ell(F_\ell(z)) H_\ell^{(\varphi, \kappa)}(z) < \gamma, \quad z \in I.$$

Then, there exists a constant $C > 0$, such that for every finite tree $T \in \mathcal{F}_M$, and every $k \geq 1$,

$$gssm_T^{(k)} \leq C \gamma^k.$$

Proof. Let T be a finite tree in \mathcal{F}_M , and let Γ, Γ' be boundary conditions defined over $L_k \cup A$, such that $\Gamma_A = \Gamma_{A'}$. As in the proof of Proposition 3.7, for $\varepsilon \in [0, 1]$, let Γ_ε be the boundary condition such that for every $v \in L_k \cup A$

$$\psi_v^{(\Gamma_\varepsilon)}(x) = (1 - \varepsilon) \psi_v^{(\Gamma)}(x) + \varepsilon \psi_v^{(\Gamma')}(x), \quad x \in X.$$

In particular we have that, for $v \in A$,

$$\psi_v^{(\Gamma_\varepsilon)}(x) = \psi_v^{(\Gamma)}(x) = \psi_v^{(\Gamma')}(x), \quad x \in X.$$

and

$$\Gamma_0 = \Gamma \text{ and } \Gamma_1 = \Gamma'.$$

Let us denote by μ_ε the measure compatible with the specification over T and the assignment Γ_ε . Also, given $v \in T$, let $\mu_{\varepsilon, v}$ be the measure compatible with the specification over $T^{(v)}$ and the restriction of the assignment Γ_ε to $T^{(v)}$. Also, denote by $\text{type}(v)$, the type of the tree subtended at v , $T^{(v)}$, in \mathcal{F}_M . Let

$$\alpha_v^{(\varepsilon)} := \mathbf{P}_{\sigma \sim \mu_{\varepsilon, v}}(\sigma(v) = 0) \quad \text{and} \quad m_v^{(\varepsilon)} := \varphi_{\text{type}(v)}(\alpha_v^{(\varepsilon)}).$$

Now, our objective is to prove that for every $v \in T$ such that $d(v, r) = i$, we have that

$$\left| \frac{\partial}{\partial \varepsilon} m_v^{(\varepsilon)} \right| \leq C' \kappa_{\text{type}(v)} \gamma^{k-i}, \quad (56)$$

for some constant C' to be determined in a moment. Notice first, that for $v \in L_{k-1} \cap A^c$ such that $\text{type}(v) = \ell$, and with children w_1, \dots, w_{b_ℓ} , we have that

$$\alpha_v^{(\varepsilon)} = 1 / \left(1 + \mathcal{J} \prod_{i=1}^{b_\ell} h(\alpha_{w_i}^{(\varepsilon)}) \right),$$

where

$$\alpha_{w_i}^{(\varepsilon)} = (1 - \varepsilon) \psi_{w_i}^{(\Gamma)}(0) + \varepsilon \psi_{w_i}^{(\Gamma')}(0).$$

Therefore,

$$\frac{\partial}{\partial \varepsilon} m_v^{(\varepsilon)} = -\mathcal{J} \left(\alpha_v^{(\varepsilon)} \right)^2 \varphi'_\ell \left(\alpha_v^{(\varepsilon)} \right) \sum_{i=1}^{b_\ell} \left(\prod_{j \neq i} h(\alpha_{w_j}^{(\varepsilon)}) \right) h' \left(\alpha_{w_i}^{(\varepsilon)} \right) \left(\psi_{w_i}^{(\Gamma')}(0) - \psi_{w_i}^{(\Gamma)}(0) \right)$$

and, using the boundedness of φ'_ℓ (for every ℓ) in I , and the boundedness of h and h' in $[0, 1]$, we get that for all $v \in L_{k-1} \cap A^c$,

$$\left| \frac{\partial}{\partial \varepsilon} m_v^{(\varepsilon)} \right| \leq C'',$$

for some constant C'' . Now, in Eq. (56), let us set $C' := \frac{C''}{\gamma_{\min_{i \in \{1, \dots, t\}} \kappa_i}}$. Then, if $v \in L_{k-1} \cap A^c$, Eq. (56) holds trivially. On the other hand, if $v \in L_{k-1} \cap A$, clearly $\frac{\partial}{\partial \varepsilon} m_v^{(\varepsilon)} = 0$, and therefore, Eq. (56) holds. Now, take $v \in L_i \cap A^c$ such that $\text{type}(v) = \ell$, with children w_1, \dots, w_{b_ℓ} such that $\text{type}(w_j) = \tau_\ell(j)$ for $j = 1, \dots, b_\ell$. We have that,

$$\alpha_v^{(\varepsilon)} = 1 / \left(1 + \mathcal{J} \prod_{i=1}^{b_\ell} h \left(\alpha_{w_i}^{(\varepsilon)} \right) \right),$$

and therefore,

$$\frac{\partial}{\partial \varepsilon} m_v^{(\varepsilon)} = \alpha_v^{(\varepsilon)} \left(1 - \alpha_v^{(\varepsilon)} \right) \varphi'_\ell \left(\alpha_v^{(\varepsilon)} \right) \sum_{j=1}^{b_\ell} \frac{H \left(\alpha_{w_j}^{(\varepsilon)} \right)}{\varphi'_{\tau_\ell(j)} \left(\alpha_{w_j}^{(\varepsilon)} \right)} \frac{\partial}{\partial \varepsilon} m_{w_j}^{(\varepsilon)},$$

Thus, assuming that (55) holds for vertices in L_{i+1} , we have that

$$\begin{aligned} \left| \frac{\partial}{\partial \varepsilon} m_v^{(\varepsilon)} \right| &\leq C' \gamma^{h-(i+1)} \alpha_v^{(\varepsilon)} \left(1 - \alpha_v^{(\varepsilon)} \right) \varphi'_\ell \left(\alpha_v^{(\varepsilon)} \right) \sum_{j=1}^{b_i} \kappa_{\text{t}(w_j)} \frac{\left| H \left(\alpha_{w_j}^{(\varepsilon)} \right) \right|}{\varphi'_{\text{type}(w_j)} \left(\alpha_{w_j}^{(\varepsilon)} \right)} \\ &\leq C' \gamma^{h-(i+1)} F_\ell(z) (1 - F_\ell(z)) \varphi'_\ell(F_\ell(z)) H_\ell^{(\varphi, \kappa)}(z) \kappa_\ell \\ &\leq C' \kappa_\ell \gamma^{h-i}, \text{ from condition 2.} \end{aligned}$$

Aso, notice that if $v \in L_i \cap A$, trivially $\left| \frac{\partial}{\partial \varepsilon} m_v^{(\varepsilon)} \right| = 0 \leq C' \kappa_{\text{type}(v)} \gamma^{h-i}$. Therefore, by induction, for all $v \in T$, Eq. (55) holds. In particular, if r is the root of T ,

$$\left| \frac{\partial}{\partial \varepsilon} m_r^{(\varepsilon)} \right| \leq C \kappa_{\text{type}(r)} \gamma^h.$$

Now,

$$\begin{aligned}
& \left| \mathbf{P}_{\sigma \sim \mu_{T, \Gamma}} (\sigma(r) = 0) - \mathbf{P}_{\sigma \sim \mu_{T, \Gamma'}} (\sigma(r) = 0) \right| \\
&= \left| \alpha_r^{(0)} - \alpha_r^{(1)} \right| \\
&\leq \int_0^1 \left| \frac{\partial}{\partial \varepsilon} \alpha_r^{(\varepsilon)} \right| d\varepsilon \\
&= \int_0^1 \left| \left(\frac{\partial}{\partial \varepsilon} m_r^{(\varepsilon)} \right) / \varphi'_{\text{type}(r)} (\alpha_r^{(\varepsilon)}) \right| d\varepsilon, \\
&\leq \frac{1}{\inf_{z \in I} \varphi'_{\text{type}(r)} (z)} \int_0^1 \left| \frac{\partial}{\partial \varepsilon} m_r^{(\varepsilon)} \right| d\varepsilon, \\
&\leq C\gamma^h, \text{ taking } C = C' \left(\max_{\ell} \kappa_{\ell} \right) / \left(\min_{\ell} \inf_{z \in I} \varphi'_{\ell} (z) \right).
\end{aligned}$$

□

The following corollary is a consequence of Proposition 3.9.

Corollary 3.14. *If the system is well oriented and the conditions of Proposition 3.13 hold then, there exists a constant C such that for every finite $T \in \mathcal{F}_{\leq M}$ and every $k \geq 1$,*

$$gssm_T^{(k)} \leq C\gamma^k.$$

3.2.3 Implications of SSM

In view of Proposition 3.6, there is a direct relation between gSSM in the path-tree and SSM in graphs. Furthermore, due to Proposition 3.9, for well oriented graphs such relation can be weakened to the approximations $\mathcal{P}_{G,v}^{(i)}$ of the path-tree. This presents an advantage, as it is the case that the structure of $\mathcal{P}_{G,v}^{(\infty)}$ presents an underlying complexity in its description, making it not suitable for a direct analysis of the SSM property (see, e.g., [67, 71, 94]). Instead, the trees $\mathcal{P}_{G,v}^{(i)}$ present a branching representation that simplifies the analysis.

We will proceed to list some standard implications of SSM for a sequence of graphs. These implications are standard and can be found in [42, 73, 114]. For further details, we redirect the reader to [96].

Following Goldberg et al. [47] we use the following variant of amenability. For $v \in V$ and a nonnegative integer d , let $B_d(v)$ denote the set of vertices within distance $\leq d$ from v .

We define the d -neighborhood amenability of the collection of graphs $\{G_n\}_{n \geq 1}$ as follows.

$$r_d = \sup_{n \geq 1} \sup_{v \in G_n} \frac{|\partial B_d(v)|}{|B_d(v)|}.$$

The collection $\{G_n\}_{n \geq 1}$ of graphs is said to be *neighborhood-amenable* if $\inf_d r_d = 0$.

Now, we can state the following theorem detailing the implications of SSM of interest to us.

Theorem 3.15. *Suppose that SSM holds for the spin system defined on the sequence of graphs $\{G_n\}_{n \geq 1}$ such that $n = |G_n|$, then*

1. *There exist $C, c > 0$ such that Weitz's algorithm [114] calculates an ϵ -approximation of the partition function $Z(G_n)$, for any $n \geq 1$, in time*

$$C (n/\epsilon)^c.$$

2. *If the family $\{G_n\}_{n \geq 1}$ is neighbourhood amenable, there exists $C > 0$ such that the Glauber dynamics in G_n , for any $n \geq 1$, mixes in Cn^2 steps.*
3. *If the family $\{G_n\}_{n \geq 1}$ is neighbourhood amenable, there exist $C > 0$ and $d > 0$, such that the d -block Glauber dynamics in G_n , for any $n \geq 1$, mixes in $Cn \log n$ steps.*
4. *If the family $\{G_n\}_{n \geq 1}$ is neighbourhood amenable, and the system is monotone², there exists $C > 0$ such that the Glauber dynamics in G_n , for any $n \geq 1$, mixes in $Cn \log n$ steps.*

For an infinite graph G , let G_n be the subgraph induced by the vertices

$$V_n = \{v \in G : d(v, v_0) \leq n\},$$

where $v_0 \in G$ is arbitrary.

Proposition 3.16. *Given a specification on G , if SSM holds for the family of graphs $\{G_n\}_{n \geq 1}$, then there is a unique Gibbs' measure on G compatible with the specification.*

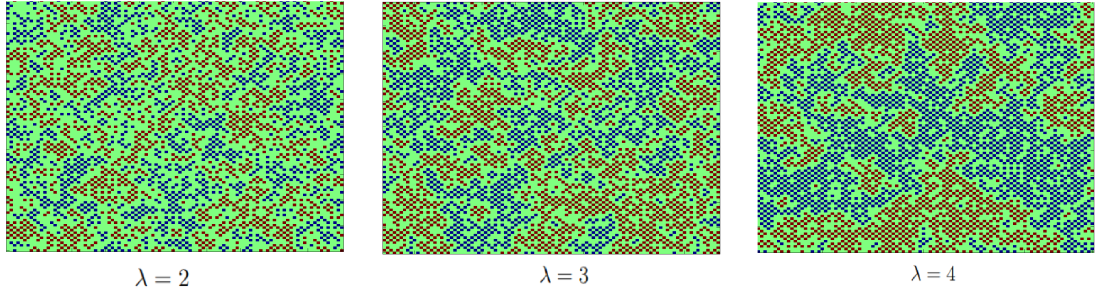


Figure 6: Random independent set in the 100×100 grid. Red dots are particles occupying even positions. Blue dots are particles occupying odd positions.

3.3 Improved uniqueness regime for the independent sets model in \mathbb{Z}^2

It is our purpose now, to obtain improved results regarding SSM and its consequences for the independent sets model in \mathbb{Z}^2 . This work builds upon the notion of path-tree of Weitz and Godsil [114, 44] introduced earlier. We focus our attention on this, which is arguably the simplest, however, not yet well-understood, case of interest. Empirical evidence suggests that the critical point at which a phase transition between uniqueness/nonuniqueness occurs is $\lambda_{\text{uniq}}(\mathbb{Z}^2) \approx 3.796$ [10, 40, 90], but rigorous results are significantly far from this conjectured point. A classical Peierls' type argument [30] implies that $\lambda_{\text{uniq}}(\mathbb{Z}^2) < C$ for some constant C . Further improvements of such argument, by Blanca et al [16] (yet unpublished), seem to put C numerically close to the conjectured threshold. On the lower bound side, Van den Berg and Steif [111] used a disagreement percolation argument to prove that $\lambda_{\text{uniq}}(\mathbb{Z}^2) > \frac{p_c}{1-p_c}$ where p_c is the critical probability for site percolation on \mathbb{Z}^2 . Applying the best known lower bound on p_c for \mathbb{Z}^2 , by Van den Berg and Ermakov [110], [111] implies $\lambda_{\text{uniq}}(\mathbb{Z}^2) > 1.255$. Prior to that work, an alternative approach aimed at establishing the Dobrushin-Shlosman criterion [32], yielded, via computer-assisted proofs, $\lambda_{\text{uniq}}(\mathbb{Z}^2) > 1.185$ by Radulescu and Styer [92], and $\lambda_{\text{uniq}}(\mathbb{Z}^2) > 1.508$ by Radulescu [91]. These results were improved by Weitz [114], who showed that $\lambda_{\text{uniq}}(\mathbb{Z}^2) \geq \lambda_{\text{uniq}}(\mathbb{T}^{(3)}) = 27/16 = 1.6875$, where $\mathbb{T}^{(3)}$ is the infinite, 3-ary tree.

In parallel to what happens with the well studied Ising model [73], it is believed that

²Recall the definition in page 6.

this threshold corresponds also to the transition between rapid mixing and exponentially slow mixing for the Glauber dynamics in \mathbb{L}_k , where \mathbb{L}_k is the subgraph of \mathbb{Z}^2 induced by the vertices at l_1 -distance $\leq k$ from the origin. Regarding this conjecture, there has been significant progress. Randall [93] proved that for $\lambda \geq 8.066$ the Glauber dynamics on \mathbb{L}_k takes exponentially many steps (on the size of \mathbb{L}_k) to mix³. On the other hand, Luby and Vigoda [68], showed that for $\lambda \leq 1$, ($\lambda < \frac{2}{\Delta-2}$ in general graphs of maximum degree Δ), the Glauber dynamics is fast, that is, there exists $C > 0$ such that for every finite subgraph G of \mathbb{Z}^2 , the Glauber dynamics mixes in $Cn \log n$ steps, where $n = |G|$. Weitz' result [114] further improves the regime of fast mixing to $\lambda < 27/16 = 1.6875$.

In this work, we present a new general approach which, for the case of the independent sets model on \mathbb{Z}^2 , improves the lower bound to $\lambda_{\text{uniq}}(\mathbb{Z}^2) > 2.3882$. There are various algorithmic implications for finite subgraphs of \mathbb{Z}^2 when $\lambda < 2.3882$. Our results imply that Weitz's deterministic FPAS is also valid on subgraphs of \mathbb{Z}^2 for such a range of λ . Thanks to the existing literature on general spin systems, our results also imply that the Glauber dynamics has $O(n \log n)$ mixing time for any finite subregion $G = (V, E)$ of \mathbb{Z}^2 (where $n = |V|$), when $\lambda < 2.3882$.

As in Weitz's work, our approach can be used for other 2-spin systems, such as the Ising model. This is discussed in Section 3.4. Our work also provides an arguably simpler way to derive the main technical result of Weitz of showing that any graph with maximum degree Δ has strong spatial mixing (SSM) when $\lambda < \lambda_{\text{uniq}}\left(T^{(\Delta-1)}\right)$. To get improved results in specific graphs, we will utilize more structural properties of self-avoiding walk trees. Weitz shows SSM in the path-tree of the graph by comparing it with $T^{(\Delta-1)}$. We refine this by comparing the path-tree with an appropriate branching tree of considerably smaller growth than $T^{(\Delta-1)}$.

3.3.1 DMS Condition: A Sufficient Criterion

In this section, we present a sufficient condition implying SSM for the independent sets model on a branching family $\mathcal{F}_{\leq M}$. In Section 3.3.2, we use such condition to establish SSM

³For the similar *torus model*, Borgs et al [17] proved that there is such slowness in the Glauber dynamics for $\lambda > 80$. Randall further improved this bound showing slowness in the torus for $\lambda > 6.183$ [93].

for finite subgraphs of \mathbb{Z}^2 . As an illustration, we will first present a condition necessary for the contraction condition from Proposition 3.7 to hold for regular trees. In particular, we reprove Weitz' result [114] that establishes SSM up to the uniqueness threshold for trees.

Let us recall the notation introduced in Section 3.2.1. For the independent sets model, we have

$$h(z) = z, F(z) = 1/(1 + \lambda z^b), I = [1/(1 + \lambda), 1].$$

Theorem 3.17. *Let $\varphi : I \rightarrow \mathbb{R}$ be the function $\varphi(z) := \frac{1}{s} \log\left(\frac{z}{s-z}\right)$, where $s = \frac{b+1}{b}$. Then, the conditions of Proposition 3.7 are satisfied whenever $\lambda < \lambda_{\text{uniq}}\left(T^{(b)}\right) = \frac{b^b}{(b-1)^{b+1}}$. In particular, SSM holds for all finite subtrees of the infinite b -ary tree $T^{(b)}$, whenever $\lambda < \lambda_{\text{uniq}}\left(T^{(b)}\right)$.*

Proof. We have that $\varphi'(x) = \frac{1}{x(s-x)}$, therefore, it is clear that φ' is bounded in I , proving condition 1 of Proposition 3.7. Now, to prove condition 2, notice first that $H(x) = \frac{\partial}{\partial x} \log h(x) = \frac{1}{x}$. Now, given $x_1, \dots, x_b \in I$ such that $\left(\prod_{i=1}^b x_i\right)^{1/b} = z$, we have that

$$\sum_{i=1}^b |H(x_i)/\varphi'(x_i)| = \sum_{i=1}^b (s - x_i) \leq b(s - z),$$

therefore, $H^{(\varphi)}(z) \leq b(s - z)$ (recall the definition of $H^{(\varphi)}$ from Eq. (54)). Therefore, it is the case that

$$F(z)(1 - F(z))\varphi'(F(z))H^{(\varphi)}(z) = \frac{b(1 - F(z))(s - z)}{s - F(z)}.$$

Now, for $g(z) := \frac{b(1-F(z))(s-z)}{s-F(z)}$, we have that $g'(z) = \frac{sb(F(z)-z)(1-F(z))}{z(s-F(z))^2}$, therefore, g reaches its maximum at the unique z^* such that $F(z^*) = z^*$. This maximum is given by $g(z^*) = b(1 - z^*) = \frac{b\omega}{1+\omega}$ (where ω is the parameter introduced for the hard-core model in example 2.1, in particular, $\omega(1 + \omega)^b = \lambda$). Therefore, when $\frac{b\omega}{1+\omega} < 1$, or equivalently, when $\lambda < \frac{b^b}{(b-1)^{b+1}}$, condition 2 from Proposition 3.7 holds. This implies SSM for the family of b -ary trees $\{T_h\}_{h \geq 1}$. Finally, it is clear that the independent sets model is well oriented. Therefore, from Proposition 3.9 the result extends to all finite subtrees of the infinite b -ary tree T . \square

The previous proposition suggests the choice of statistic $\varphi_\ell(z) = \frac{1}{s_\ell} \log\left(\frac{z}{s_\ell - z}\right)$, with

appropriate parameters s_j , in the general case of branching trees. In fact, under this choice, we obtain the following condition for SSM.

Definition 3.18 (DMS Condition). *Given a $t \times t$ branching matrix M and $\lambda^* > 0$, for $s_1, \dots, s_t > 1$ and $\kappa = (\kappa_1, \dots, \kappa_t) > 0$, let D and S be the diagonal matrices defined as*

$$D_{\ell,\ell} = \sup_{z \in I} \frac{(1-z) \left(1 - \theta_\ell \left(\frac{1-z}{\lambda^* z}\right)^{1/b_\ell}\right)}{s_\ell - z} \quad \text{and} \quad S_{\ell,\ell} = s_\ell,$$

where

$$\theta_\ell := \frac{\left(\prod_{j=1}^t \kappa_j^{M_{\ell,j}}\right)^{1/b_\ell}}{\sum_{j=1}^t \kappa_j s_j M_{\ell,j}/b_\ell} \quad \text{and} \quad b_\ell = \sum_{j=1}^t M_{\ell,j}.$$

We say the DMS Condition holds for M and λ^* , if there exist $s_1, \dots, s_t > 1$ and $\kappa > 0$ such that:

$$(DMS) \quad \kappa < \kappa. \quad (57)$$

Notice, in particular, that this condition is monotone in λ . That is, if DMS holds for M and λ^* , then DMS holds for M and every $\lambda < \lambda^*$.

Theorem 3.19. *If the DMS Condition holds for M and $\lambda^* > 0$, then the conditions for contraction from Proposition 3.13 hold with the choice of statistics $\varphi_\ell(z) = \frac{1}{s_\ell} \log \left(\frac{z}{s_\ell - z}\right)$ for $\ell = 1, \dots, t$. Consequently, there is SSM for the family of trees $\mathcal{F}_{\leq M}$ for any $\lambda \leq \lambda^*$. That is, for every $\lambda \leq \lambda^*$, there exist $C > 0$ and $\gamma \in (0, 1)$ such that for every finite tree $T \in \mathcal{F}_{\leq M}$,*

$$ssm_T^{(k)} \leq C \gamma^k, \quad k \geq 1.$$

Proof. It is clear that the functions $\{\varphi_\ell\}_{\ell=1}^t$ are bounded in $I = \left[\frac{1}{1+\lambda}, 1\right]$, therefore, condition 1 from Proposition 3.13 holds. Now, given $x_1, \dots, x_{b_\ell} \in I$ such that $\left(\prod_{j=1}^{b_\ell} x_j\right)^{1/b_\ell} = z$, we have that

$$\sum_{j=1}^{b_\ell} \left| H(x_j) / \varphi'_{\tau_\ell(j)}(x_j) \right| \kappa_{\tau_i(j)} = \sum_{j=1}^{b_\ell} (s_j - x_j) \kappa_{\tau_i(j)} \leq \sum_{j=1}^{b_\ell} s_j \kappa_{\tau_i(j)} - b_\ell z \left(\prod_{j=1}^{b_\ell} \kappa_{\tau_i(j)} \right)^{1/b_\ell},$$

therefore, $H_\ell^{(\varphi, \kappa)}(z) \leq \frac{1}{\kappa_\ell} (1 - z \theta_\ell) \left(\sum_{j=1}^{b_\ell} s_j \kappa_{\tau_i(j)} \right)$. Now,

$$\begin{aligned} F(z) (1 - F(z)) \left| \varphi'_\ell(F(z)) \right| H_\ell^{(\varphi, \kappa)}(z) &\leq \frac{(1 - F(z)) (1 - \theta_\ell z)}{(s_\ell - F(z))} \frac{1}{\kappa_\ell} \left(\sum_{j=1}^{b_\ell} s_j \kappa_{\tau_i(j)} \right) \\ &\leq \frac{D_{\ell,\ell}}{\kappa_\ell} \left(\sum_{j=1}^{b_\ell} s_j \kappa_{\tau_i(j)} \right). \end{aligned}$$

Therefore, if Eq. (57) holds, it is the case that

$$D_{\ell,\ell} \left(\sum_{j=1}^{b_\ell} s_j \kappa_{\tau_i(j)} / b_\ell \right) < \kappa_\ell.$$

In particular, there exists $\gamma \in (0, 1)$ such that for all $\ell \in \{1, \dots, t\}$,

$$F(z) (1 - F(z)) |\varphi'_\ell(F(z))| H_\ell^{(\varphi, \kappa)}(z) \leq \frac{D_{\ell,\ell}}{\kappa_\ell} \left(\sum_{j=1}^{b_\ell} s_j \kappa_{\tau_i(j)} / b_\ell \right) < \gamma.$$

Thus, condition 2 is satisfied. Therefore, SSM holds for the family \mathcal{F}_M . The result extends to $\mathcal{F}_{\leq M}$ by means of Proposition 3.9 (due to the fact that the independent sets model is well oriented). \square

3.3.2 Application of the DMS condition

Now, we will show how the use of Theorem 3.19 leads to an improvement in the lower bound on $\lambda_{\text{uniq}}(\mathbb{Z}^2)$. Throughout the section, we will discuss the proof of the following theorem.

Theorem 3.20. *There exists a $t \times t$ branching matrix M such that*

- *For every finite subgraph G of \mathbb{Z}^2 and every $v \in G$, $\mathcal{P}_v^{(\infty)}(G) \in \mathcal{F}_{\leq M}$.*
- *The DMS condition holds for M , for $\lambda^* = 2.3882$.*

The following corollary is a consequence of the previous Theorem, in view of Theorems 3.19 and 3.15.

Corollary 3.21. *The following holds for \mathbb{Z}^2 , for all $\lambda \leq \lambda^* = 2.3882$.*

1. *SSM holds for finite subgraphs of \mathbb{Z}^2 . That is, there exist $C > 0$ and $\gamma \in (0, 1)$ such that for every finite subgraph G of \mathbb{Z}^2 , every $v \in G$ and every $k \geq 1$,*

$$ssm_{G,v}^{(k)} \leq C\gamma^k.$$

2. *There exists a **unique** infinite-volume Gibbs' measure for the independent sets model in \mathbb{Z}^2 .*
3. *For any $\lambda \leq \lambda^*$, there exist constants $C, c > 0$, such that Weitz' algorithm [114] calculates an ϵ -approximation of the partition function $Z(G)$ for any $G \subseteq \mathbb{Z}^2$, in time $C(n/\epsilon)^c$, where $n = |G|$.*

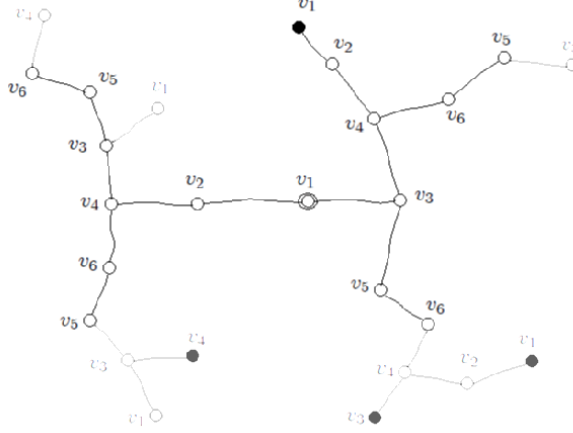


Figure 7: Reduction of the path-tree $\mathcal{P}_v^{(\infty)}(G)$ to $\tilde{\mathcal{P}}_v^{(\infty)}(G)$ for the graph in Figure 3. The shaded vertices are the ones deleted in the reduction.

4. For any $\lambda \leq \lambda^*$, there exists $C > 0$, such that the Glauber dynamics mixes in $Cn \log n$ iterations for any finite subgraph G of \mathbb{Z}^2 , where $n = |G|$.

To prove Theorem 3.20, first we show that, for every $i < \infty$, there exists $M^{(i)}$ such that for every finite subgraph G of \mathbb{Z}^2 , and every $v \in G$, $\mathcal{P}_v^{(i)}(G) \in \mathcal{F}_{\leq M^{(i)}}$. This implies, in particular, that $\mathcal{P}_v^{(\infty)}(G) \in \mathcal{F}_{\leq M^{(i)}}$. The matrix $M^{(i)}$ can be regarded as an i^{th} approximation of the tree $\mathcal{P}_v^{(\infty)}(G)$. Then, in view of Theorem 3.19, we proceed to calculate a value λ^* such that the DMS condition holds for $M^{(i)}$. This is enough to prove Theorem 3.20. However, there is room for improvement by taking into account the discrete assignment that is imposed to the lifted measure defined on $\mathcal{P}_v^{(i)}(G)$ (see Eq. (46)). In the particular case of the hard-core model, such discrete assignment can be realized, equivalently, by considering the ‘chopped’ tree $\tilde{\mathcal{P}}_v^{(i)}(G)$. This is defined as the subtree of $\mathcal{P}_v^{(i)}(G)$ generated by the vertices (i.e. paths) $p \in \mathcal{P}_v^{(i)}(G)$ such that

- p is not terminal.
- p does not have a terminal child that is positively oriented.

Analogously to Eq. (45), it is clear that $\tilde{\mathcal{P}}_v^{(1)}(G) \supseteq \tilde{\mathcal{P}}_v^{(2)}(G) \supseteq \dots \supseteq \tilde{\mathcal{P}}_v^{(\infty)}(G)$. Moreover, if we denote by μ_G the Gibbs’ measure defined on G and by $\tilde{\mu}_T$ the Gibbs’ measure

defined over $\widetilde{\mathcal{P}}_v^{(\infty)}(G)$, from Theorem 3.4 is clear that

$$\mathbf{P}_{\sigma \sim \mu}(\sigma(v) = x) = \mathbf{P}_{\sigma \sim \widetilde{\mu}_{(\infty)}}(\sigma_r = x).$$

Therefore, instead, we look for matrices $\widetilde{M}^{(i)}$ such that $\widetilde{\mathcal{P}}_v^{(i)}(G) \in \mathcal{F}_{\leq \widetilde{M}^{(i)}}$, and then we proceed to verify the DMS condition for such $\widetilde{M}^{(i)}$.

The definitions of $M^{(i)}$ and $\widetilde{M}^{(i)}$ are not too complicated. Their construction is reminiscent of the strategy employed in [7, 88] to give an upper bound on the connectivity constant of several lattice graphs, including \mathbb{Z}^2 .

In the following, we denote by o the origin in \mathbb{Z}^2 . In order to define $M^{(i)}$, let $\mathcal{S}^{(i)} := \{p \in \mathcal{P}_o^{(i)}(\mathbb{Z}^2) : l(p) \leq i\}$. Now, we define $M^{(i)}$ to be the matrix indexed by the elements of $\mathcal{S}^{(i)}$, and such that for any $p, p' \in \mathcal{S}^{(i)}$ ⁴,

$$M_{p,p'}^{(i)} = \begin{cases} 1 & \text{if } p' \text{ is the augmentation of } p \text{ (in } \mathcal{P}_v^{(i)}) \\ 0 & \text{otherwise} \end{cases}.$$

Proposition 3.22. *It is the case that $\mathcal{P}_o^{(i)}(\mathbb{Z}^2) \in \mathcal{F}_{\leq M^{(i)}}$. In particular, for every subgraph G of \mathbb{Z}^2 , and every $v \in G$, $\mathcal{P}_v^{(i)}(G) \in \mathcal{F}_{\leq M^{(i)}}$.*

Proof. Define the function $Q : \mathcal{P}_o^{(i)}(\mathbb{Z}^2) \rightarrow \mathcal{S}^{(i)}$ as follows,

$$Q(p) := \begin{cases} p & \text{if } l(p) \leq i \\ v_{k-i}, \dots, v_k & \text{if } p = v_0, \dots, v_k, \text{ where } k > i. \end{cases}.$$

Then, a straightforward induction implies that for any finite subtree T of $\mathcal{P}_o^{(i)}(\mathbb{Z}^2)$, T is a tree of type $Q(p)$ in $\mathcal{F}_{\leq M^{(i)}}$. The second statement follows from the fact that every $p \in \mathcal{P}_v^{(i)}(G)$ is the translation of a path $p' \in \mathcal{P}_o^{(i)}(\mathbb{Z}^2)$. \square

Similarly, let $\widetilde{\mathcal{S}}^{(i)} := \{p \in \widetilde{\mathcal{P}}_o^{(i)}(\mathbb{Z}^2) : l(p) \leq i\}$. Now, define $\widetilde{M}^{(i)}$ to be the matrix indexed by the elements of $\widetilde{\mathcal{S}}^{(i)}$, and such that for any two $p, p' \in \widetilde{\mathcal{S}}^{(i)}$,

$$\widetilde{M}_{p,p'}^{(i)} = \begin{cases} 1 & \text{if } p' \text{ is the augmentation of } p \text{ (in } \widetilde{\mathcal{P}}_v^{(i)}) \\ 0 & \text{otherwise} \end{cases}.$$

The proof of the following proposition is analogous to Proposition 3.22.

⁴Recall the definition of ‘augmentation’ from page 54.

Table 1: Improved uniqueness threshold for the Independent sets model on \mathbb{Z}^2 .

Approximation	λ^*
$M^{(4)}$	1.8801
$\widetilde{M}^{(4)}$	2.1625
$\widetilde{M}^{(6)}$	2.3335
$\widetilde{M}^{(8)}$	2.3882

Proposition 3.23. *It is the case that $\widetilde{\mathcal{P}}_o^{(i)}(\mathbb{Z}^2) \in \mathcal{F}_{\leq \widetilde{M}^{(i)}}$. In particular, for every subgraph G of \mathbb{Z}^2 , and every $v \in G$, $\widetilde{\mathcal{P}}_v^{(i)}(G) \in \mathcal{F}_{\leq \widetilde{M}^{(i)}}$.*

The task of checking the DMS condition is almost purely computational (although **provable**). Table 1 summarizes the threshold λ^* we obtain for each matrix.

For any such matrix, the verification of the DMS Condition relies on (i) ‘guessing’ appropriate values for the parameters S and κ and (ii) formally verifying that the DMS Condition holds for the chosen S and κ . In choosing adequate S and κ , we employ a heuristic algorithm. On the other hand, to verify that the DMS Condition holds for a given *rational* matrix S and vector κ is straightforward, provided we can obtain a rational upper bound for each type ℓ , for the function:

$$f_\ell(\alpha) = \frac{(1 - \alpha) \left(1 - \theta_\ell \left(\frac{1 - \alpha}{\lambda \alpha} \right)^{1/\Delta_j} \right)}{s_\ell - \alpha}.$$

Indeed, due to the concavity of this function for $0 < \theta_\ell \leq 1$, $s_\ell > 51/50$ and $\lambda > 27/16$ ⁵, it is always possible to find a *provable* upper bound for f_ℓ in such a regime. This can be done, for example, by prescribing a suitable ‘envelope’ for f_ℓ consisting of a piecewise linear function of the form:

$$g_\ell(\alpha) = \begin{cases} B_\ell & \text{if } \alpha < \alpha_\ell \\ \min\{b_\ell(\alpha - \alpha_\ell) + B_\ell, b_u(\alpha - \alpha_u) + B_u\} & \text{if } \alpha_\ell < \alpha < \alpha_u \\ B_u & \text{if } \alpha > \alpha_u \end{cases}$$

where $\alpha_\ell, \alpha_u \in [0, 1]$ are such that $b_\ell > f'_j(\alpha_\ell) > 0$, $b_u < f'_j(\alpha_u) < 0$, $B_\ell > f_j(\alpha_\ell)$ and $B_u > f_j(\alpha_u)$. It is clear for any such function, that $g_\ell(\alpha) > f_\ell(\alpha)$, thus we obtain a provable upper bound for f_ℓ using g_ℓ .

⁵This is a nontrivial, although merely algebraic, fact. It can be proved using a standard symbolic

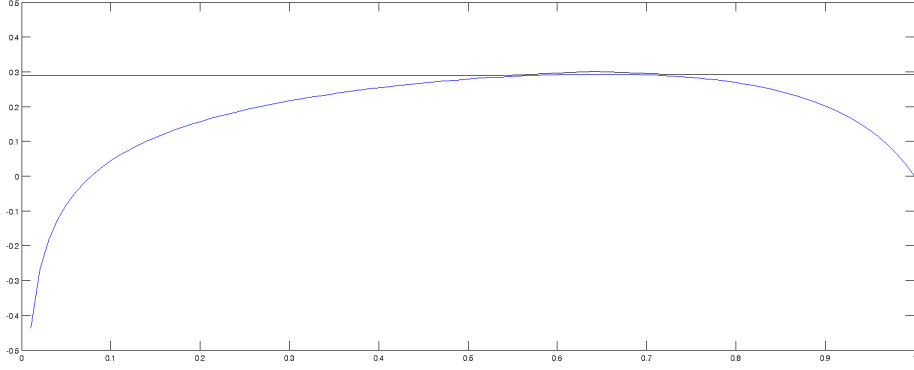


Figure 8: The function f and its envelope, for $\lambda = 2$, $s = 1.2$, $\theta = 0.7$ and $d = 5$.

For every matrix in the above table, the values of S and κ , along with appropriate envelopes that lead to upper bounds $\hat{D}_{\ell,\ell}$ for the corresponding $D_{\ell,\ell}$, are included in the online appendix [95]. In particular, for $M^{(4)}$, the following matrix gives an equivalent branching:

$$\mathbf{N} = \begin{pmatrix} 0 & 4 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$

where the type $\ell = 0, \dots, 3$ of a vertex (path) in the tree represents the fact that a continuation with a minimum of $4 - \ell$ additional edges is needed to complete a cycle of length 4. For this branching matrix, one can check that the (DMS) condition of Theorem 3.19 holds with $\lambda^* = 1.8801$, $\mathbf{S} = \mathbf{Diag}(1.040, 1.388, 1.353, 1.255)$ and $\kappa = (0.266037, 0.100891, 0.100115, 0.0973861)$.

3.4 A further example: The Ising Model

The approach taken for the independent sets model can be used to address corresponding questions in the well-studied Ising model with inverse temperature β and external field \mathcal{J} . In this case, the appropriate statistic to be used for a branching matrix M is

$$\varphi_\ell(z) = \varphi(z) := \log \left(\frac{z}{1-z} \right), \quad \ell \in \{1, \dots, t\}$$

computation. For further details, we redirect the reader to our article [96].

Notice that, in this case, $h(z) = \frac{e^\beta - (e^\beta - e^{-\beta})z}{(e^\beta - e^{-\beta})z + e^{-\beta}}$, and therefore,

$$\begin{aligned} |H(z)/\varphi'(z)| &= \frac{|e^{2\beta} - e^{-2\beta}| z(1-z)}{[|e^\beta - e^{-\beta}| z + e^{-\beta}][e^\beta - |e^\beta - e^{-\beta}| z]} \\ &= \frac{|e^{2\beta} - e^{-2\beta}| u}{[e^\beta + ue^{-\beta}][e^\beta u + e^{-\beta}]}, \quad \text{where } u = (1-z)/z \\ &\leq |\tanh \beta| \quad (\text{the previous expression reaches its maximum at } u = 1). \end{aligned}$$

This implies that for any given κ ,

$$H_\ell^{(\varphi, \kappa)}(z) \leq \frac{1}{\kappa_\ell} |\tanh \beta| \sum_{j=1}^t M_{\ell,j} \kappa_j, \quad \ell \in \{1, \dots, b\}.$$

To verify conditions 1 and 2 from Proposition 3.13, notice first that $I = \left[\frac{e^{-\beta}}{e^\beta + e^{-\beta}}, \frac{e^\beta}{e^\beta + e^{-\beta}} \right]$, so that clearly φ_ℓ is bounded on I . On the other hand, let us take, in particular, κ to be the right eigenvector associated with the maximum eigenvalue θ of the matrix M . Notice that, due to Perron-Frobenius theorem, κ is positive. Now,

$$F(z)(1-F(z))\varphi'_\ell(F(z))H_\ell^{(\varphi, \kappa)}(z) \leq \frac{1}{\kappa_\ell} |\tanh \beta| \sum_{j=1}^t M_{\ell,j} \kappa_j = \theta |\tanh \beta|.$$

Then, provided $\theta |\tanh \beta| < 1$, condition 2 holds. Therefore, we conclude:

Theorem 3.24. *Given a branching matrix M , let \mathcal{G}_M be the family of graphs G such that for every $v \in G$, $\mathcal{P}_v^{(\infty)}(G) \in \mathcal{F}_{\leq M}$. If $\theta := \text{Spec}(M)$ then, for the Ising model with arbitrary external field \mathcal{J} and inverse temperature β such that $|\beta| < \tanh^{-1}(1/\theta)$, SSM holds for the family \mathcal{G}_M . That is, there exist $C > 0$ and $\gamma \in (0, 1)$ ⁶ such that for every finite graph $G \in \mathcal{G}_M$, and every $v \in G$,*

$$ssm_{G,v}^{(k)} \leq C\gamma^k, \quad k \geq 1.$$

In particular, the conclusions of Theorem 3.15 hold for the family of graphs \mathcal{G}_M .

In particular, for the Ising model in \mathbb{Z}^d , SSM holds for all $\beta < \beta^*$, as detailed in the table 2.

In comparison, applying Weitz's general technique to \mathbb{Z}^2 implies SSM for $|\beta| < .34657$. We do not investigate the Ising model further because there are much stronger results known

⁶An noteworthy observation is that C and γ do not depend on \mathcal{J} . (However, they may depend on β and M).

Table 2: Lower bounds for the uniqueness threshold using the path-tree approach, for the Ising model on \mathbb{Z}^d .

Dimension	β^*
2	0.392190
3	0.214247
4	0.148045
5	0.113347

for this model. Onsager [84] established that $\beta_{\text{uniq}}(\mathbb{Z}^2) = \log(1 + \sqrt{2}) \approx 0.440686$. And, for general trees, Lyons [70, Theorem 2.1] established the critical point for uniqueness, which coincides with ours, guaranteeing that the statistic chosen is optimal.

In fact, the Ising model covers, by an easy translation among the models (see, e.g. [100, 46]), all the ‘soft’ 2-spin systems with **arbitrary** self-interactions. Let us assume that $\phi = \begin{bmatrix} e^\alpha & e^\delta \\ e^\delta & e^\beta \end{bmatrix}$ and ψ , the self-interaction, is arbitrary. Then, there is SSM for the family $\mathcal{G}_{\leq M}$ provided

$$\left| \tanh \left(\frac{\alpha + \beta - 2\delta}{4} \right) \right| \text{Spec}(M) < 1.$$

CHAPTER IV

RANDOM GEOMETRIES

We now take a detour to the case of random graph structures or ‘mean field’ case. Paradoxically, for this, seemingly complex case, there is a wide availability of tools for the study of phase transitions provided by the extra-randomness. In this chapter, our objective is to study the decay of correlation for the q -coloring model on sparse random graphs. For this goal, we make recurrent use of different tools and results available for this model.

Our interest in this model lies in the fact that random structures provide an outstanding example of phase transitions in statistical physics and combinatorics. They render one of the ‘natural’ situations in which an underlying ‘hardness’ appears when the density of the random graph increases. For instance, for a big class of constraint satisfaction problems (among others, random k -SAT and random graph/hypergraph coloring), all known polynomial time algorithms stop finding solutions at much smaller densities than the one at which the support of the Gibbs’ measure becomes empty. For example, it is well-known that it is easy to color a random graph using twice as many colors as its chromatic number, by using a simple greedy strategy. On the other hand, yet to date, no algorithm is known that uses less than twice the chromatic number of the graph. In fact, the factor of 2 corresponds in a precise mathematical sense to a phase transition in the geometry of the support of the Gibbs’ measure for the q -coloring model, called *clustering* [1]. For other models like k -SAT, independent sets [26], or a more general class of CSP’s defined in [78], the situation is similar: at such a ‘clustering’ threshold, the model becomes, as far as the current knowledge, computationally ‘hard’.

By the same token, several research directions [1, 59, 77], both heuristic and rigorous, motivated two possible explanations for the failure of polynomial algorithms: **(1)** The space of solutions becomes increasingly complex as the number of constraints increases and is not captured correctly by simple algorithms (that is, the clustering explanation stated above).

(2) Typical solutions become increasingly correlated and local algorithms cannot unveil such correlations. Our aim in this chapter is to study the existence of such ‘correlations’ for the q -coloring model, providing further evidence for the conjectural ‘hardness’.

This ‘correlation’ problem was studied in some detail in the context of Gibbs measures on trees [15, 81, 101]. For random graphs, as we do here, Montanari and Gerschenfeld [43] initiated a program, in which under a ‘sphericity’ condition, they obtain a direct translation of correlation rates between the random graph and a corresponding Poisson random tree. We take such a direction for our study.

In the following, for every $n \geq 1$, V_n is a set with n elements. The *random graph* $G(n, \alpha)$ is the graph with vertex set V_n and such that we choose every possible edge, independently, to be in the graph with probability α/n . The parameter α is called the *density* of the graph. This construction corresponds to the classical Erdős-Rényi random graph model [36] in its ‘sparse’ regime.

As a matter of notation, the symbols \mathbb{P} and \mathbb{E} refer to probability and expectation, with respect to the random graph $G(n, \alpha)$. We say that a graph property P holds *with high probability* (w.h.p.), if

$$\lim_{n \rightarrow \infty} \mathbb{P}(G(n, \alpha) \text{ satisfies } P) = 1.$$

Likewise, we say that the property holds *with positive probability* (w.p.p), if

$$\liminf_{n \rightarrow \infty} \mathbb{P}(G(n, \alpha) \text{ satisfies } P) > 0.$$

For every $n \geq 1$, let $\mu_{n, \alpha}$ be the Gibbs’ measure corresponding to a given specification in $G(n, \alpha)$ (in our case, the specification is the q -coloring model). In the following, \mathbf{v} is a vertex chosen uniformly at random in V_n .

We say that there is *correlation* in the model, if there exists a sequence $\{\epsilon_k\}_{k \geq 1}$ with $\limsup_{k \rightarrow \infty} \epsilon_k > 0$ such that, with positive probability, for every $k \geq 1$,¹

$$I_{G_n}^{(k)}(\mathbf{v}) \geq \epsilon_k.$$

Further, we say that there is *vanishing correlation* in the model, if there exists a sequence

¹Recall the definition of $I_{G_n}^{(k)}(v)$ from page 11 in Chapter 1.

$\{\epsilon_k\}_{k \geq 1}$ with $\lim_{k \rightarrow \infty} \epsilon_k = 0$ such that, with high probability, for every $k \geq 1$,

$$I_{G_n}^{(k)}(\mathbf{v}) \leq \epsilon_k.$$

The main result of this chapter consists in showing that, for the q -coloring model in $G(n, \alpha)$, correlations appear in a region that coincides with the conjectural ‘hard’ regime. More precisely,

Theorem 4.1. *Let us consider the (random) Gibbs’ measure for the q -coloring specification in the random graph $G(n, \alpha)$.*

1. *For every $\delta > 0$, there exists $q_0(\delta)$ such that for all $q > q_0(\delta)$, if $\alpha = (1 + \delta)q \log q$, the q -coloring model in $G(n, \alpha)$ exhibits correlation.*
2. *For every $\delta > 0$, there exists $q_0(\delta)$ such that for all $q > q_0(\delta)$, if $\alpha = (1 - \delta)q \log q$, the q -coloring model in $G(n, \alpha)$ has vanishing correlation.*

We establish a similar result for a bigger class of models in the research article [78]. The proof of the theorem is contained in Section 4.1.

Gerschenfeld - Montanari correspondence: Notice that the random graph $G(n, \alpha)$ has typically $\approx \alpha n/2$ edges. Therefore, the average degree is, roughly, α . In this regime, (called also sparse regime), the graph is characterized by the emergence of a giant connected component for $\alpha > 1$. One of the reasons why this is called *sparse* is because, for every $k \geq 1$, if \mathbf{v} is a vertex chosen uniformly at random in V_n then, w.h.p, the graph induced by $L_{\leq k}(\mathbf{v})$ is a Galton-Watson tree of depth k with offspring distribution $\text{Poisson}(\alpha)$.

For the q -coloring model in $G(n, \alpha)$, Achlioptas and Naor [3] established that the chromatic number is $\approx \frac{\alpha}{2 \log \alpha}$ ². In particular, they proved that for $\alpha > 2q \log q$, w.h.p the graph $G(n, \alpha)$ is not q -colorable, while for $\alpha < 2(q - 1) \log(q - 1)$, it is q -colorable w.h.p..

Our result relies on a connection between reconstruction in random graphs and Poisson trees established by Gerschenfeld and Montanari in [43] that we describe next.

²More precisely they proved that, w.h.p., the chromatic number of $G(n, \alpha)$ is either q_α or $q_\alpha + 1$ where q is the smallest integer such that $\alpha < 2q \log q$.

Recall that we use \mathbf{P} and \mathbf{E} to denote probability and expectation respect to the Gibbs' measure $\mu_{n,\alpha}$. Instead, we use the symbols \mathbb{P} and \mathbb{E} for calculations regarding the randomness of $G(n, \alpha)$.

Let Ω_n denote the set of assignments $\eta : V_n \rightarrow [q]$. For a given assignment $\eta \in \Omega_n$, we define

$$m_\eta(x) := \# \frac{1}{n} \{v \in V_n : \eta(v) = x\}, \quad x \in X.$$

And, for $\eta, \eta' \in \Omega_n$, we define

$$m_{\eta, \eta'}(x, y) := \# \frac{1}{n} \{v \in V_n : \eta(v) = x \text{ and } \eta'(v) = y\}, \quad x, y \in X.$$

The *balance* of $\mu_{n,\alpha}$ is defined as

$$\mathcal{A}(\mu_{n,\alpha}) := \mathbf{E}_{\sigma \sim \mu_{n,\alpha}} \left[\sum_{x \in X} |m_\sigma(x) - 1/q| \right],$$

and the *discrepancy* of $\mu_{n,\alpha}$ is defined as

$$\mathcal{D}(\mu_{n,\alpha}) := \mathbf{E}_{(\sigma, \sigma') \sim \mu_{n,\alpha}^{(2)}} \left[\sum_{x, y \in X} |m_{\sigma, \sigma'}(x, y) - 1/q^2| \right].$$

Here, $\mu_{n,\alpha}^{(2)} := \mu_{n,\alpha} \otimes \mu_{n,\alpha}$ is the uniform measure over pairs of proper colorings of the graph.

We denote by $T(\alpha)$, the infinite Galton-Watson tree with offspring distribution Poisson(α). Let us use, for the moment, the symbol \mathbb{P} to denote probability respect to the randomness of $T(\alpha)$. We say that there is *correlation* in the random tree $T(\alpha)$, if

$$\mathbb{P} \left(\limsup_{k \rightarrow \infty} I_{T(\alpha)}^{(k)} > 0 \right) > 0.$$

On the other hand, we say that there is *vanishing correlation* in the random tree $T(\alpha)$, if

$$\mathbb{P} \left(\lim_{k \rightarrow \infty} I_{T(\alpha)}^{(k)} = 0 \right) = 1.$$

The following is the correspondence established by Montanari and Gerschenfeld:

Theorem 4.2 ([43, Theorem 1.4]). *If the discrepancy of $\mu_{n,\alpha}$ vanishes, that is,*

$$\lim_{n \rightarrow \infty} \mathbb{E}[\mathcal{D}(\mu_{n,\alpha})] = 0.$$

Then, the following holds:

1. If there is correlation in $T(\alpha)$ then, there is correlation in $G(n, \alpha)$.
2. If there is vanishing correlation in $T(\alpha)$ then, there is vanishing correlation in $G(n, \alpha)$.

Clustering. Given two assignments $\eta, \eta' \in \Omega_n$, the *Hamming distance* between η and η' is defined as

$$d_H(\eta, \eta') := \sum_{v \in V_n} \mathbf{I}[\eta(v) \neq \eta'(v)].$$

Also, for two sets $A, B \subseteq \Omega_n$, their Hamming distance is defined as

$$d_H(A, B) := \min_{\eta \in A, \eta' \in B} d_H(\eta, \eta').$$

Let $S(n, \alpha) := \text{supp } \mu_{n, \alpha}$, that is, the assignments $\eta : V_n \rightarrow [q]$ that are proper colorings of $G(n, \alpha)$. We say that $S(n, \alpha)$ *shatters* (or, exhibits *clustering*), if there exist constants $\beta, \gamma, \zeta > 0$ such that, w.h.p., $S(n, \alpha)$ can be partitioned into disjoint regions so that:

- The number of regions is at least $e^{\beta n}$.
- Each region contains at most an $e^{-\gamma n}$ fraction of the elements of $S(n, \alpha)$.
- The Hamming distance between any two regions is at least ζn .

The ‘clustering’ theorem of Achlioptas and Coja-Oghlan states that the space of solutions shatters for an appropriate regime of the density α :

Theorem 4.3 ([1]). *For every $\delta > 0$, there exists $q_0(\delta)$ such that, for every $q \geq q_0(\delta)$, the space of q -colorings of the random graph $G(n, \alpha)$ (that is, $S(n, \alpha)$) shatters, w.h.p., for*

$$(1 + \delta) q \ln q \leq \alpha \leq (2 - \delta) q \ln q.$$

Our main result in this chapter (Theorem 4.1) states that, in the clustering regime, there is also correlation in the model. This supports further the existence of an underlying ‘hardness’ for the coloring model in such a regime, providing an additional reason (besides clustering) for the failure of polynomial algorithms at such density values.

4.1 Correlation in the coloring model: Proof of Theorem 4.1

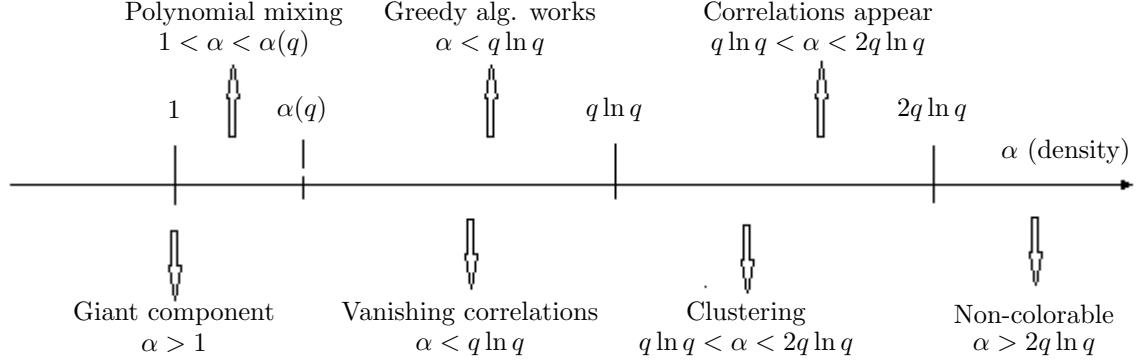


Figure 9: Phase transition for the q -coloring model in the random graph $G(n, \alpha)$

Now, we proceed to prove Theorem 4.1. The proof relies on the verification of the condition in Theorem 4.2 for the q -coloring model. Once we establish the condition, that is, that the discrepancy vanishes for the coloring model, Theorem 4.1 follows by recalling, from [15, 101], the following result for $T(\alpha)$ ³:

Theorem 4.4 ([15, 101]). *For every $\delta > 0$, there exists $q_0(\delta)$ such that for every $q \geq q_0(\delta)$:*

1. *There is correlation in the Poisson tree $T(\alpha)$, if $\alpha > (1 + \delta) q \log q$.*
2. *There is vanishing correlation in the Poisson tree $T(\alpha)$, if $\alpha < (1 - \delta) q \log q$.*

In order to verify the condition in Theorem 4.2, we need some preliminary lemmas and notation. First of all, we recall the following estimates from [3, 1].

Lemma 4.5. *Let $Z_{n,\alpha}$ be the partition function for the q -coloring model in $G(n, \alpha)$, and let $\bar{Z}_{n,\alpha} := \mathbb{E}[Z_{n,\alpha}]$.*

³These results are actually straightforward extensions of the corresponding arguments for non-reconstruction (in [101]) and reconstruction (in [15])

1. [1, Lemma 7]. If $\alpha < q \log q$, there exists a function $f(n)$ with $\lim_{n \rightarrow \infty} f(n)/n = 0$ such that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(Z_{n,\alpha} < e^{-f(n)} \bar{Z}_{n,\alpha} \right) = 0. \quad (58)$$

2. [1]. If $\alpha < q \log q$, the balance of the measure $\mu_{n,\alpha}$ vanishes, that is,

$$\lim_{n \rightarrow \infty} \mathbb{E} [\mathcal{A}(\mu_{n,\alpha})] = 0. \quad (59)$$

3. [3]. For every $n \geq 1$,

$$\bar{Z}_{n,\alpha} \geq \frac{C_{\alpha,q}}{n^{(q-1)/2}} \left[q \left(1 - \frac{1}{q} \right)^\alpha \right]^n, \quad (60)$$

where $C_{\alpha,q}$ is a fixed constant depending on α and q , but independent of n .

Now, let us introduce some notation. If m is a $q \times q$ positive matrix, let \mathcal{H} and \mathcal{E} denote the *entropy* and *energy* of m , respectively, where

$$\begin{aligned} \mathcal{H}(m) &= - \sum_{i,j} m(i,j) \log m(i,j) \\ \mathcal{E}(m) &= \log \left(1 - \sum_i \left(\sum_j m(i,j) \right)^2 - \sum_j \left(\sum_i m(i,j) \right)^2 + \sum_{i,j} m(i,j)^2 \right). \end{aligned}$$

Also, let \bar{m} be the matrix with all entries $1/q^2$. Thus, in particular, $\mathcal{H}(\bar{m}) = \log q^2$, and $\mathcal{E}(\bar{m}) = 2 \log(1 - 1/q)$.

Given $\epsilon, \delta > 0$, let $\mathcal{M}(\delta, \epsilon)$ denote the set of all $q \times q$ matrices m with nonnegative entries such that

$$\|(m - \bar{m}) \mathbf{1}\|^2 \leq \delta, \quad \|\mathbf{1}^t (m - \bar{m})\|^2 \leq \delta \quad \text{and} \quad \|m - \bar{m}\|^2 \geq \epsilon,$$

where $\mathbf{1}$ is the $q \times 1$ vector of all 1's and $\|\cdot\|$ is the Euclidean l_2 -norm (for matrices or vectors).

Now we present some estimates concerning an additive functional depending on the energy and entropy of matrices in $\mathcal{M}(\delta, \epsilon)$. For this purpose, we define $\kappa(\delta, \epsilon)$ as the upper limit of the interval (indeed, easy to see that this is an interval) consisting of the values c such that

$$\sup_{v \in \mathcal{M}(\delta, \epsilon)} \mathcal{H}(v) + c \mathcal{E}(v) \leq \mathcal{H}(\bar{v}) + \alpha \mathcal{E}(\bar{v}).$$

To motivate, let us recall that an important part of the second moment argument of Achlioptas and Naor [3, Theorem 7] (in showing that the chromatic number of $G(n, \alpha)$ is concentrated on two possible values), relied on an optimization of the expression $\mathcal{H}(v) + \alpha\mathcal{E}(v)$ over the Birkhoff polytope $\mathcal{B}_{q \times q}$ consisting of the set of $q \times q$ doubly stochastic matrices. In particular, they proved that, as long as $\alpha \leq (q-1) \log(q-1)$, one has

$$\sup_{v \in \mathcal{B}_{q \times q}} \mathcal{H}(v) + \alpha\mathcal{E}(v) = \mathcal{H}(\bar{m}) + \alpha\mathcal{E}(\bar{m}). \quad (61)$$

In particular, since $\mathcal{M}(0, \epsilon) \subseteq \mathcal{B}_{q \times q}$, we have $\kappa(0, \epsilon) \geq \alpha_q = (q-1) \log(q-1)$. This implies also, due to the continuity of $\kappa(\delta, \epsilon)$, that whenever $\alpha < \alpha_q$, for every $\epsilon > 0$ there is some $\delta > 0$ such that $\kappa(\delta, \epsilon) > \alpha$.

Lemma 4.6. *Suppose that $m \in \mathcal{M}(\delta, \epsilon)$ where $\epsilon > 2\delta$. Then, if $\kappa(\delta, \epsilon) > \alpha$, we have that*

$$[\mathcal{H}(m) + \alpha\mathcal{E}(m)] \leq [\mathcal{H}(\bar{m}) + \alpha\mathcal{E}(\bar{m})] - \frac{(\kappa(\delta, \epsilon) - \alpha)(\epsilon - 2\delta)}{2(1 - 1/q)^2}.$$

Proof. Indeed,

$$\begin{aligned} & [\mathcal{H}(\bar{m}) + \alpha\mathcal{E}(\bar{m})] - [\mathcal{H}(m) + \alpha\mathcal{E}(m)] \\ &= [\mathcal{H}(\bar{m}) + \kappa(\delta, \epsilon)\mathcal{E}(\bar{m})] - [\mathcal{H}(m) + \kappa(\delta, \epsilon)\mathcal{E}(m)] + (\kappa(\delta, \epsilon) - \alpha)[\mathcal{E}(m) - \mathcal{E}(\bar{m})] \\ &\geq (\kappa(\delta, \epsilon) - \alpha) \left[\log \left(1 + \frac{1}{(1 - 1/q)^2} \left[\|m - \bar{m}\|^2 - \|(m - \bar{m})\mathbf{1}\|^2 - \|\mathbf{1}^t(m - \bar{m})\|^2 \right] \right) \right] \\ &\geq \frac{(\kappa(\delta, \epsilon) - \alpha)(\epsilon - 2\delta)}{2(1 - 1/q)^2}. \end{aligned}$$

□

Now, let $Z^{(2)} := Z \otimes Z$ and $\mu^{(2)} := \mu \otimes \mu$, that is, the counting measure and the uniform measure, respectively, over pairs of colorings of the graph.

Given $\epsilon > 0$, choose $\delta < \epsilon/2$, such that $\kappa(\delta, \epsilon) > \alpha$ (see the comment prior to Lemma 4.6), and let $\xi = \frac{(\kappa(\delta, \epsilon) - \alpha)(\epsilon - 2\delta)}{2(1 - 1/q)^2}$. We have that

$$\mathbf{P}_{(\sigma, \sigma') \sim \mu^{(2)}} \left(\|m_{\sigma, \sigma'} - \bar{m}\|^2 > \epsilon \right) = \frac{Z^{(2)} \left(\left\{ (\eta, \eta') : \|m_{\eta, \eta'} - \bar{m}\|^2 > \epsilon \right\} \right)}{Z^2}.$$

Now, according to Eqs. (58) and (59), the events $Z < e^{-n\xi} \bar{Z}$, $\|(m_{\sigma, \sigma'} - \bar{m})\mathbf{1}\|^2 > \epsilon$ and $\|\mathbf{1}^t(m_{\sigma, \sigma'} - \bar{m})\|^2 > \epsilon$ are ‘negligible’, that is:

- $\lim_{n \rightarrow \infty} \mathbb{P} (Z < e^{-n\xi} \bar{Z}) = 0.$
- $\lim_{n \rightarrow \infty} \mathbb{E} \left[\mathbf{P}_{(\sigma, \sigma') \sim \mu^{(2)}} \left(\left\| (m_{\sigma, \sigma'} - \bar{m}) \mathbf{1} \right\|^2 > \epsilon \right) \right] = 0.$
- $\lim_{n \rightarrow \infty} \mathbb{E} \left[\mathbf{P}_{(\sigma, \sigma') \sim \mu^{(2)}} \left(\left\| \mathbf{1}^t (m_{\sigma, \sigma'} - \bar{m}) \right\|^2 > \epsilon \right) \right] = 0.$

It is our aim to prove that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\mathbf{P}_{(\sigma, \sigma') \sim \mu^{(2)}} \left(\left\| m_{\sigma, \sigma'} - \bar{m} \right\|^2 > \epsilon \right) \right] = 0. \quad (62)$$

In view of the ‘negligibility’ mentioned above, it is sufficient to prove that

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E} \left[Z^{(2)} \left(\left\{ (\eta, \eta') : m_{\eta, \eta'} \in \mathcal{B}_{q \times q}^{\delta, \epsilon} \right\} \right) \right]}{(e^{-n\xi} \bar{Z})^2} = 0. \quad (63)$$

Now, consider the set $\mathcal{G}_{\epsilon, \delta}$ of $q \times q$ matrices L , with nonnegative integer entries, such that $L/n \in \mathcal{M}(\delta, \epsilon)$. Also, denote by R_m the set of pairs of colorings (not necessarily proper) (η, η') such that $m_{\eta, \eta'} = m$. Then,

$$\begin{aligned} & \mathbb{E} \left[Z^{(2)} \left(\left\{ (\eta, \eta') : m_{\eta, \eta'} \in \mathcal{B}_{q \times q}^{\delta, \epsilon} \right\} \right) \right] \\ &= \sum_{L \in \mathcal{G}_{\epsilon, \delta}} \sum_{\eta, \eta' \in R_{L/n}} \mathbb{P} (\eta \text{ and } \eta' \text{ are proper colorings of } G(n, \alpha)) \\ &= \sum_{L \in \mathcal{G}_{\epsilon, \delta}} \frac{n!}{\prod_{i,j} L_{ij}!} \left[\frac{n}{n-1} \right]^{\alpha n} \left(1 - \sum_i \left(\sum_j L_{ij}/n \right)^2 - \sum_j \left(\sum_i L_{ij}/n \right)^2 + \sum_{i,j} (L_{ij}/n)^2 \right)^{\alpha n} \\ &\leq \sum_{L \in \mathcal{G}_{\epsilon, \delta}} 3q^{2q} \sqrt{n} \exp(n [\mathcal{H}(L/n) + \alpha \mathcal{E}(L/n)]). \end{aligned}$$

We can invoke Lemma 4.6 to get that

$$[\mathcal{H}(L/n) + \alpha \mathcal{E}(L/n)] \leq [\mathcal{H}(\bar{m}) + \alpha \mathcal{E}(\bar{m})] - \xi.$$

Therefore,

$$\mathbb{E} \left[Z^{(2)} \left(\left\{ (\eta, \eta') : m_{\eta, \eta'} \in \mathcal{B}_{q \times q}^{\delta, \epsilon} \right\} \right) \right] \leq \text{poly}(n) \times [q(1 - 1/q)^\alpha]^{2n} \exp(-n\xi),$$

and, applying Eq. (60), we have

$$\frac{\mathbb{E} \left[Z^{(2)} \left(\left\{ (\eta, \eta') : m_{\eta, \eta'} \in \mathcal{B}_{q \times q}^{\delta, \epsilon} \right\} \right) \right]}{(e^{-n\xi} \bar{Z})^2} \leq \text{poly}(n) \times \exp(-n\xi).$$

Therefore, Eq. (63) holds and, subsequently, Eq. (62). The condition in Theorem 4.2 follows, since, for any $\epsilon > 0$,

$$\mathcal{D}(\mu) \leq \epsilon + \mathbf{P}_{(\sigma, \sigma') \sim \mu^{(2)}} \left(\|m_{\sigma, \sigma'} - \bar{m}\|^2 > \epsilon \right).$$

Therefore, for every $\epsilon > 0$, $\limsup_{n \rightarrow \infty} \mathbb{E}[\mathcal{D}(\mu)] \leq \epsilon$, i.e., $\lim_{n \rightarrow \infty} \mathbb{E}[\mathcal{D}(\mu)] = 0$. This finishes our proof.

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